Brief Review of Fluctuation Measures

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RHIC/AGS User's Meeting

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Topics

- Charge fluctuations
- Charge correlations
- $\langle p_{\mathrm{t}} \rangle$ fluctuations
- p_{t} correlations
- Linearity and locality
- Correlations from fluctuations
- Fluctuation measure design

Introduction

- Many fluctuation measures can be constructed from a basic set of random variables
- How are they related?
- Is there a 'best' set of measures?
- How can results be interpreted?

$\equiv$ means algebraically equivalent for all n

$\approx$ means approximation, which may be very bad for small n

Measure criteria:

1) simple reference form for *central limit* (CLT) *conditions*
2) independent of sample number for CLT conditions
3) 'good behavior' for small sample numbers
4) simple linear relation to two-particle correlations

Summary of Charge Fluctuation Measures

$$Variance\ \langle \delta X^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2$$

$$v(Q) \equiv \langle \delta Q^2 \rangle / \langle N_{CH} \rangle$$

$$N_{CH} = N_+ + N_-$$

$$R = \frac{\langle N_+ \rangle}{\langle N_- \rangle}$$

$$Q = N_+ - N_-$$

PHENIX

STAR

$$D \equiv \langle N_{ch} \rangle \langle \delta R^2 \rangle = 4 \frac{\langle \delta Q^2 \rangle}{\langle N_{CH} \rangle}$$

$$v(Q) \approx 1 + \frac{\langle N_+ + N_- \rangle}{4} \nu_{+-,dyn}$$

$$D \approx 4 + \langle N_+ + N_- \rangle \nu_{+-,dyn}$$

$$D = 4v(Q)$$

$$\nu_{+-,dyn} = \nu_{+-} - \nu_{+-,stat}$$

$$\nu_{+-} = \left\langle \left(\frac{N_+}{\langle N_+ \rangle} - \frac{N_-}{\langle N_- \rangle} \right)^2 \right\rangle$$

$$\overline{z^2} = 4 \frac{\langle N_+ \rangle \langle N_- \rangle}{\langle N_{CH}{}^2 \rangle}$$

$$\Phi_q \approx \frac{\langle N_+ \rangle^{3/2} \langle N_- \rangle^{3/2}}{\langle N_{CH} \rangle^2} \nu_{+-,dyn}$$

$$\nu_{+-,stat} = \frac{1}{\langle N_+ \rangle} + \frac{1}{\langle N_- \rangle}$$

$$Z = Q - \frac{\langle Q \rangle}{\langle N_{CH} \rangle} N_{CH}$$

NA49

$$\Gamma = v(Q)$$

$$\Phi_q = \sqrt{\frac{\langle Z^2 \rangle}{\langle N_{CH} \rangle}} - \sqrt{\overline{z^2}}$$

Jeff Mitchell summary

$$\Gamma \equiv \frac{1}{\langle N_{CH} \rangle} \left\langle \left(Q - \frac{\langle Q \rangle}{\langle N_{CH} \rangle} N_{CH} \right)^2 \right\rangle$$

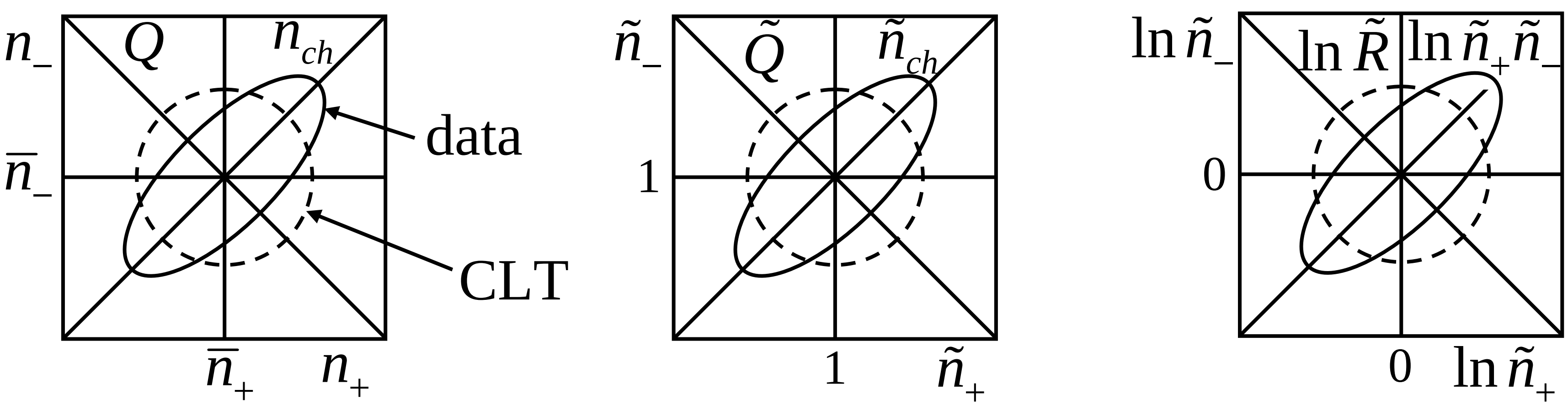
Charge Fluctuations Basics

$n_+(\delta x),\ n_-(\delta x)$, basic random variables in bins of size (scale) δx

$n_{ch} \equiv n_+ + n_-,\ Q \equiv n_+ - n_-,\ \ R \equiv n_+ / n_-$ sum, difference, ratio

specific variables based on $\tilde{n}_+ \equiv n_+ / \bar{n}_+,\ \tilde{n}_- \equiv n_- / \bar{n}_-$

$\tilde{n}_{ch} \equiv \tilde{n}_+ + \tilde{n}_-,\ \tilde{Q} \equiv \tilde{n}_+ - \tilde{n}_-,\ \ \tilde{R} \equiv \tilde{n}_+ / \tilde{n}_-$ $\qquad Q \leftrightarrow n_\Delta \quad n_{\mathrm{ch}} \leftrightarrow n_\Sigma$



problem: what are variances on Q and n_{ch} *relative to* CLT

Number Variances and the CLT

$\sigma_X^2(\delta x) \equiv \left\langle \delta X^2(\delta x) \right\rangle \equiv \overline{\left\{ X(\delta x) - \bar{X}(\delta x) \right\}^2}$ variance of *rv* X

$\sigma_{n_{ch}}^2, \quad \sigma_Q^2, \quad \sigma_{\tilde{Q}}^2 \equiv \nu_{+-}, \quad \sigma_R^2 \approx \sigma_{\tilde{Q}}^2$ variances of derived *rv*

$\sigma_{n_{ch}}^2 / \bar{n}_{ch}, \ \sigma_Q^2 / \bar{n}_{ch} \equiv \nu(Q), \ \bar{n}_{ch} \sigma_R^2 \equiv D$ normalized variances

normalized variance: *per particle*

$$\bar{R} = \bar{n}_+ / \bar{n}_- \left(1 + \underbrace{\sigma_{n_-}^2 \Big/ 2\bar{n}_-^2 + \cdots}_{\textbf{problems for small } n_{\mathrm{ch}}} \right)$$

CLT reference:

central limit theorem: *independent samples from fixed parent*

$$\sigma_{n_{ch}}^2 / \bar{n}_{ch} \to 1, \ \sigma_Q^2 / \bar{n}_{ch} \to 1, \ \sigma_R^2 \approx \sigma_{\tilde{Q}}^2 \to 1/\bar{n}_+ + 1/\bar{n}_- \equiv \nu_{+-,\mathrm{stat}}$$

$$\bar{n}_{ch} \sigma_{\tilde{Q}}^2 \equiv \bar{n}_{ch} \nu_{+-} \approx \bar{n}_{ch} \sigma_R^2 \equiv D \to \approx 4$$

Differential Fluctuation Measures

$$\Delta\sigma^2_{n_{ch}/} \equiv \sigma^2_{n_{ch}} / \bar{n}_{ch} - 1, \;\; \Delta\sigma^2_{Q/} \equiv \sigma^2_Q / \bar{n}_{ch} - 1 \equiv \nu(Q) - 1$$

$$\nu_{+-,\text{dyn}} \equiv \nu_{+-} - \nu_{+-,\text{stat}}, \quad D-4 \qquad \text{data} - \text{CLT}$$

$$\sigma^2_{n_{ch}} \equiv \sigma^2_+ + 2\sigma^2_{+-} + \sigma^2_-$$

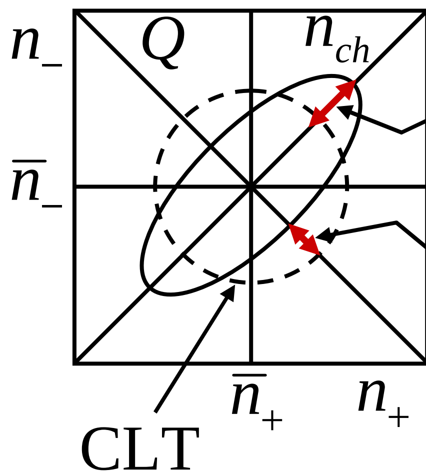
$$\sigma^2_Q \equiv \sigma^2_+ - 2\sigma^2_{+-} + \sigma^2_-$$

Pearson's corr. coefficient

$$\frac{\Delta\sigma^2_{n_{ch}/} - \Delta\sigma^2_{Q/}}{2} \equiv \frac{2\left(\overline{n_+ n_-} - \bar{n}_+ \bar{n}_-\right)}{\left(\bar{n}_+ + \bar{n}_-\right)}$$

$$\approx \frac{\sigma^2_{n_{ch}} - \sigma^2_Q}{\sigma^2_{n_{ch}} + \sigma^2_Q} \equiv \frac{\sigma^2_{+-}}{\left(\sigma^2_+ + \sigma^2_-\right)/2} \equiv r_{+-}$$

normalized charge covariance



$$\Delta\sigma^2_{n_{ch}/} \equiv \sigma^2_{n_{ch}} / \bar{n}_{ch} - 1$$

$$\Delta\sigma^2_{Q/} \equiv \nu(Q) - 1$$

$$\approx D/4 - 1$$

$$\approx \bar{n}_{ch}\, \nu_{+-,\text{dyn}} / 4$$

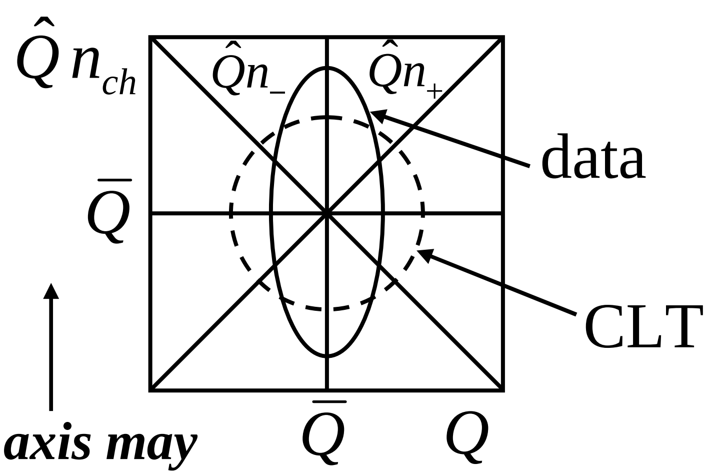
this plot varies with bin size δx: scale dependence $\leftrightarrow$ correlations

Φ_q

$$\Phi_q \equiv \sqrt{\overline{Z_q^2} / \bar{n}_{ch}} - \hat{z}_q, \quad Z_q \equiv Q - n_{ch}\hat{Q}, \quad \hat{Q} \equiv \bar{Q} / \bar{n}_{ch}$$

$$\overline{Z_q^2} \equiv \sigma_Q^2 - 2\hat{Q}\sigma_{Qn_{ch}}^2 + \hat{Q}^2\sigma_{n_{ch}}^2 \quad \rightarrow \bar{n}_{ch}(1 - \hat{Q}^2) \text{ (CLT)}$$

$$\hat{z}_q^2 \equiv 1 - \hat{Q}^2, \qquad \sigma_{Qn_{ch}}^2 \equiv \sigma_+^2 - \sigma_-^2 \quad \rightarrow \bar{n}_{ch}\hat{Q} \text{ (CLT)}$$



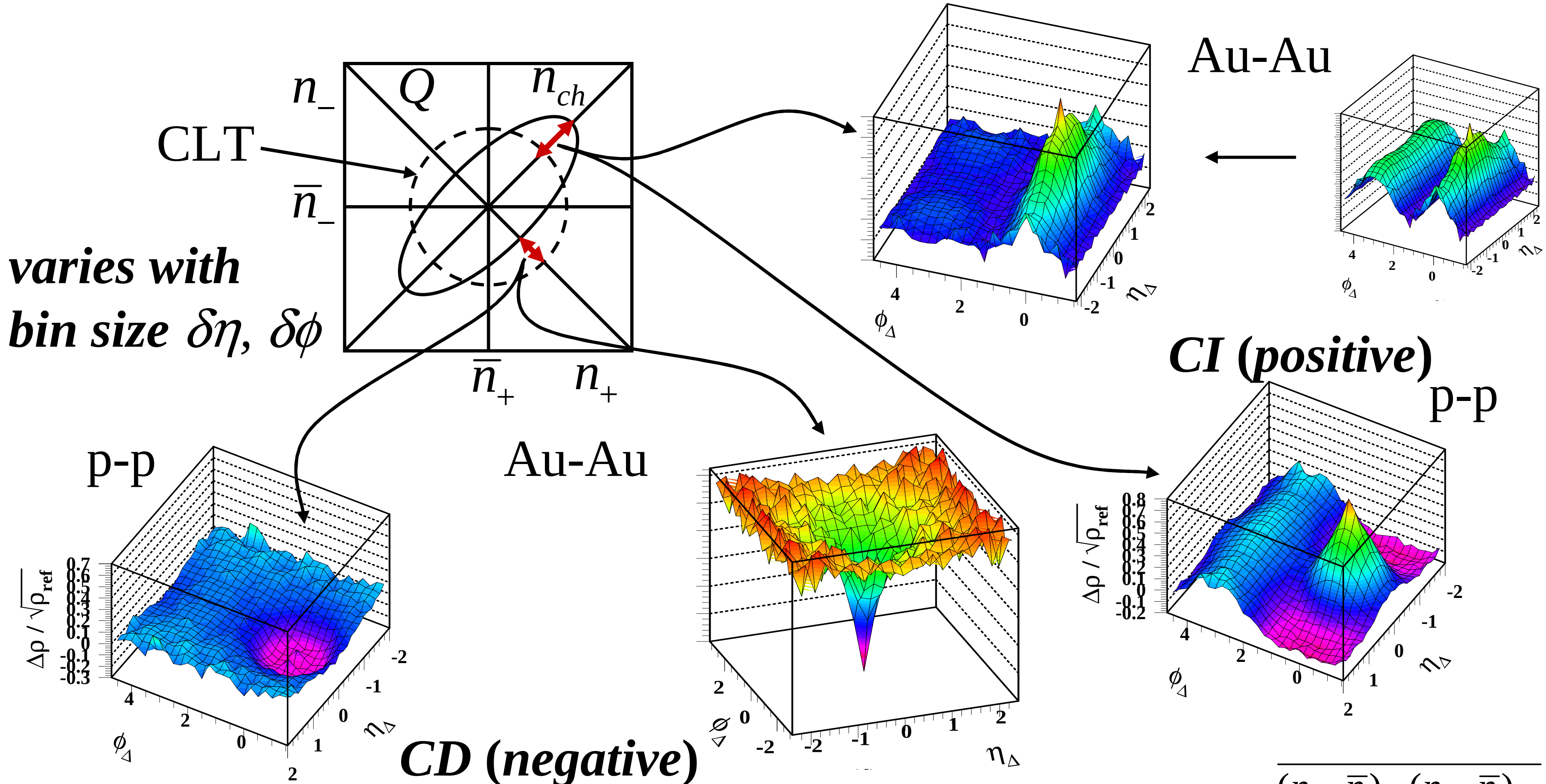
this axis may have zero extent

$$\Delta\sigma_{Q/}^2 \equiv \sigma_Q^2 / \bar{n}_{ch} - 1 \equiv \nu(Q) - 1$$

$$\Phi_q \approx \sqrt{\sigma_Q^2 / \bar{n}_{ch}} - 1$$

this measure rotates the (n_+,n_-) space by 45° – nothing further is gained

What's Behind the *n* Fluctuations?



fluctuation scale (bin size) dependence
integrates these two-particle correlations

$$\Delta\rho / \sqrt{\rho_{ref}}\ \Big|_{abcd} \equiv \frac{\overline{(n-\bar{n})_{ab}(n-\bar{n})_{cd}}}{\varepsilon_\eta \varepsilon_\phi \sqrt{\bar{n}_{ab}\,\bar{n}_{cd}}}$$

on $(\eta_1, \eta_2, \phi_1, \phi_2)$

Report Card – Number *n* Fluctuations

Measure criteria:

1) simple reference form for *central limit* (CLT) *conditions*
2) independent of sample number *n* for CLT conditions
3) 'good behavior' for small sample number
4) simple linear relation to two-particle correlations

measure	simple ref. CLT form	CLT form indep. of *n*	good for small *n*	simple rel. to two-pt.
$\Delta\sigma^2_{nch}$	yes	yes	yes	yes
$\Delta\sigma^2_{Q}$	yes	yes	yes	yes
$\nu(Q)$	yes	yes	yes	yes
$\nu_{+-,dyn}$	yes	no	?	no
Φ_q	no	~	yes	no
D	no	no	no	no

Summary of Event-by-event <p_T> Fluctuation Measures

Goal of the observables:

State a comparison to the expectation of statistically independent particle emission.

Jeff Mitchell summary

F_{pT} PHENIX

$$F_{p_T} \approx \frac{\Phi_{p_T}}{\sigma_{p_T,incl.}}$$

$\sigma^2_{pT,dyn}$ STAR

$$\sigma^2_{p_T,dyn} \cong \frac{2\Phi_{p_T}\sqrt{\overline{\Delta p_T^2}}}{\langle N \rangle}$$

$\Delta\sigma_{pT,n}$ STAR

$$\Delta\sigma_{p_T,n} \cong \sqrt{\left(\Phi_{p_T} + \sigma_{p_T,incl}\right)^2 - \sigma^2_{p_T,incl}}$$

$$\sigma_{p_T,incl.} = \sqrt{\langle p_T^2 \rangle - \langle p_T \rangle^2}$$

$$\overline{\Delta p_T^2} \equiv \overline{p_T^2} - \overline{p_T}^2$$

$$\Sigma_{p_T} = \frac{\sigma_{p_T}}{\overline{\overline{p_T}}}\sqrt{\frac{2F_{p_T}}{< N >}}$$

$$\Sigma_{p_T} \equiv \operatorname{sgn}\left(\sigma^2_{p_T,dyn}\right)\frac{\sqrt{|\sigma^2_{p_T,dyn}|}}{\overline{p}_T}$$

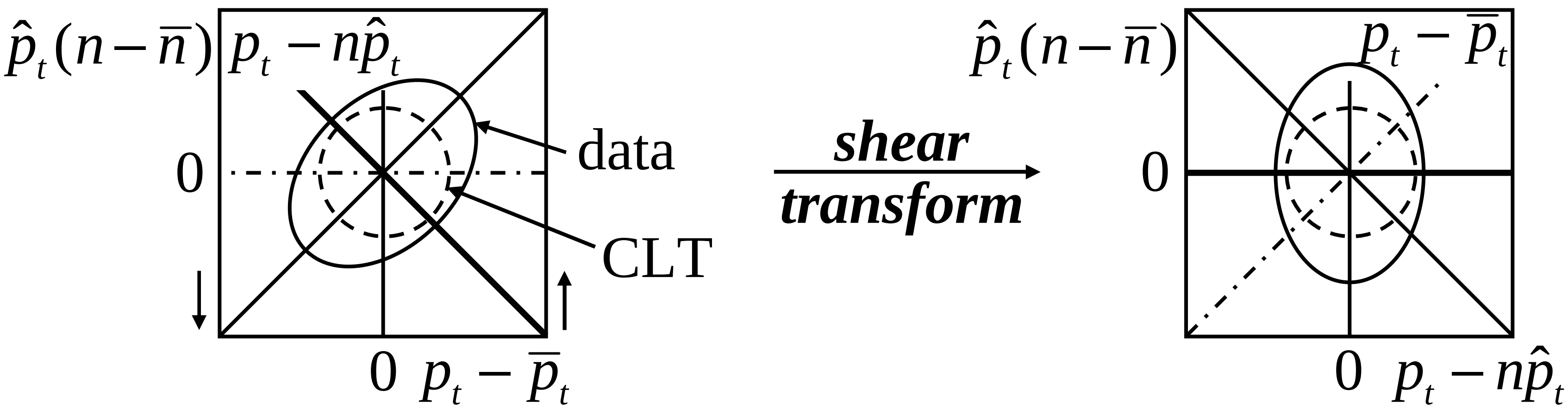
Φ_{pT} NA49

Σ_{pT} CERES

$\langle p_t \rangle$ Fluctuations Basics

$p_t(\delta x)$, $n(\delta x)$, basic random variables in bins of size (scale) δx

$p_t - n\,\hat{p}_t$, $\quad R \rightarrow \langle p_t \rangle \equiv p_t / n$ difference, ratio; $\quad \hat{p}_t \equiv \overline{p}_t / \overline{n}$



$$p_t - \overline{p}_t = p_t - n\hat{p}_t + \hat{p}_t(n - \overline{n})$$

problem: what is variance on $p_t - n\,\hat{p}_t$ *relative to* CLT

p_t Variances and the CLT

$$\overline{(p_t - n\,\hat{p}_t)^2},\quad \hat{p}_t^2\overline{(n-\bar{n})^2},\quad \sigma^2_{\langle p_t\rangle} \equiv \overline{\left(\langle p_t\rangle - \hat{p}_t\right)^2}$$ var. for derived *rv*

$$\sigma^2_{p_t:n} \equiv \overline{(p_t - n\,\hat{p}_t)^2}/\bar{n},\quad \hat{p}_t^2\overline{(n-\bar{n})^2}/\bar{n},\quad \bar{n}\sigma^2_{\langle p_t\rangle}$$ normalized var.

$$\hat{p}_t\,\overline{(p_t - n\,\hat{p}_t)(n-\bar{n})}/\bar{n}$$ normalized covariance

decomposition of p_t variance

$$\overline{\left(p_t - \bar{p}_t\right)^2} = \overline{\left(p_t - n\hat{p}_t\right)^2} + 2\hat{p}_t\overline{\left(p_t - n\hat{p}_t\right)(n-\bar{n})} + \hat{p}_t^2\overline{(n-\bar{n})^2}$$

CLT reference:

independent samples from fixed parent p_t spectrum

$$\overline{(p_t - n\,\hat{p}_t)^2} \rightarrow \bar{n}\sigma^2_{\hat{p}_t},\quad \hat{p}_t^2\overline{(n-\bar{n})^2} \rightarrow \bar{n}\,\hat{p}_t^2$$

Differential Fluctuation Measures

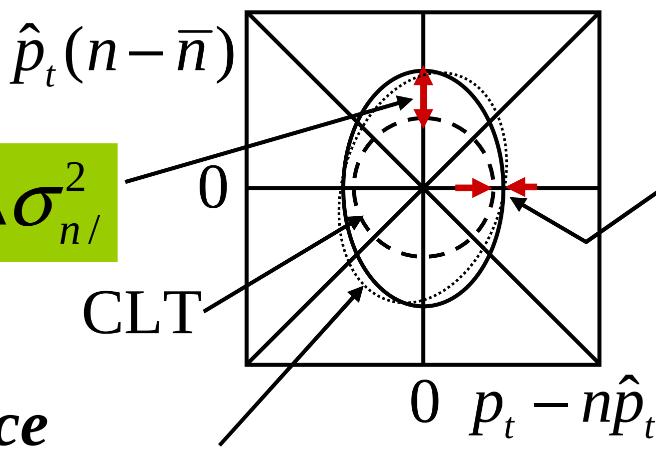
$$\Delta\sigma^2_{p_t:n}(\delta x) \equiv \sigma^2_{p_t:n}(\delta x) - \sigma^2_{\hat{p}_t}, \quad \Delta\sigma^2_{n/} \equiv \overline{(n-\bar{n})^2}/\bar{n} - 1$$

$$\Phi_{p_t} \equiv \sigma_{p_t:n} - \sigma_{\hat{p}_t} \approx \Delta\sigma_{p_t:n}, \quad rms \text{ version of } \Delta\sigma^2_{p_t:n}$$

$$\left\langle \delta p_{ti}\delta p_{tj} \right\rangle_{i\neq j} \equiv \overline{(p_t - n\,\hat{p}_t)^2/\{n(n-1)\}} - \sigma^2_{\hat{p}_t}\overline{1/(n-1)}$$

$$\bar{n}\sigma^2_{\langle p_t\rangle} - \sigma^2_{\hat{p}_t}$$

$$\Delta\sigma^2_{n/}$$



$$\Delta\sigma^2_{p_t:n}, \ \Phi_{p_t}, \left\langle \delta p_{ti}\delta p_{tj} \right\rangle_{i\neq j}$$

$$\Sigma_{p_t} \equiv \sqrt{\left\langle \delta p_{ti}\delta p_{tj} \right\rangle / \hat{p}_t^2}$$

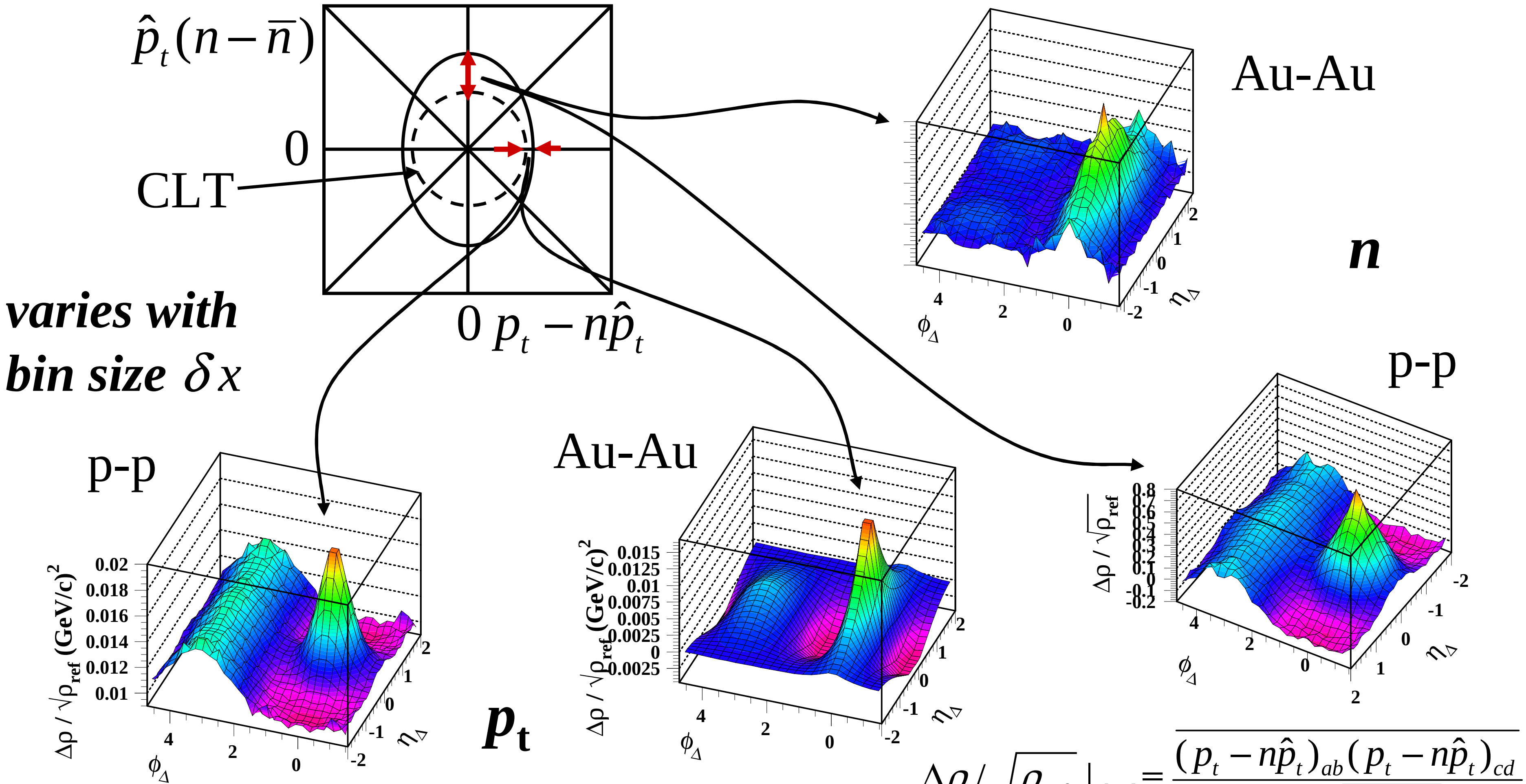
$$F_{p_t} \equiv \sigma_{\langle p_t\rangle,data} / \sigma_{\langle p_t\rangle,mix} - 1$$

p_t:n-n covariance

$$\hat{p}_t\,\overline{(p_t - n\,\hat{p}_t)(n-\bar{n})}/\bar{n}$$

this plot varies with bin size δx:
scale dependence $\leftrightarrow$ correlations

What's Behind the $\langle\, p_t \,\rangle$ Fluctuations?



$$\Delta\rho / \sqrt{\rho_{ref}}\ |_{abcd} \equiv \frac{\overline{(p_t - n\hat{p}_t)_{ab}(p_t - n\hat{p}_t)_{cd}}}{\varepsilon_\eta \varepsilon_\phi \sqrt{\bar{n}_{ab}\,\bar{n}_{cd}}}$$

on $(\eta_1, \eta_2, \phi_1, \phi_2)$

fluctuation scale (bin size) dependence integrates these two-particle correlations

Report Card – $\langle p_t \rangle$ Fluctuations

Measure criteria:

1) simple reference form for *central limit* (CLT) *conditions*
2) independent of sample number *n* for CLT conditions
3) 'good behavior' for small sample number
4) simple linear relation to two-particle correlations

measure	simple ref. CLT form	CLT form indep. of *n*	good for small *n*	simple rel. to two-pt.
$\Delta\sigma^2_{pt:n}$	yes	yes	yes	yes
Φ_{pt}	yes	yes	yes	no
F_{pt}	no	no	no	no
$\langle \delta p_t \cdot \delta p_t \rangle$	no	no	no	no
Σ_{pt}	no	no	no	no

Linearity and Locality

***composite system*:**

$$n_{tot} = n_a + n_b$$

a	b
n_a	n_b

$$r_{ab} = \frac{\overline{(n-\bar{n})_a (n-\bar{n})_b}}{\sqrt{\overline{(n-\bar{n})_a^2} \cdot \overline{(n-\bar{n})_b^2}}}$$

$$\sigma^2_{n_{tot}/} \equiv \bar{n}_a / \bar{n}_{tot} \cdot \sigma_a^2 / \bar{n}_a + 2\sqrt{\bar{n}_a \bar{n}_b} / \bar{n}_{tot} \cdot \sigma_{ab}^2 / \sqrt{\bar{n}_a \bar{n}_b} + \bar{n}_b / \bar{n}_{tot} \cdot \sigma_b^2 / \bar{n}_b$$

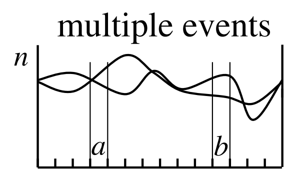
$$\sigma_{ab}^2 / \sqrt{\bar{n}_a \bar{n}_b}$$

normalized covariance represents a-b correlation

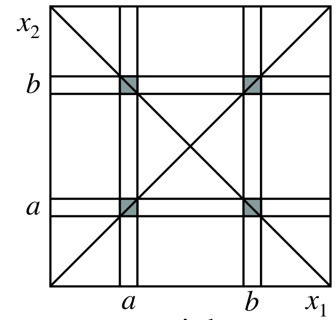
$$\rightarrow \frac{\overline{(n-\bar{n})_a (n-\bar{n})_b}}{\sqrt{\bar{n}_a \bar{n}_b}}$$

$$r_{aa} \rightarrow \frac{\overline{(n-\bar{n})^2}}{\bar{n}} \equiv \sigma^2_{n/}$$

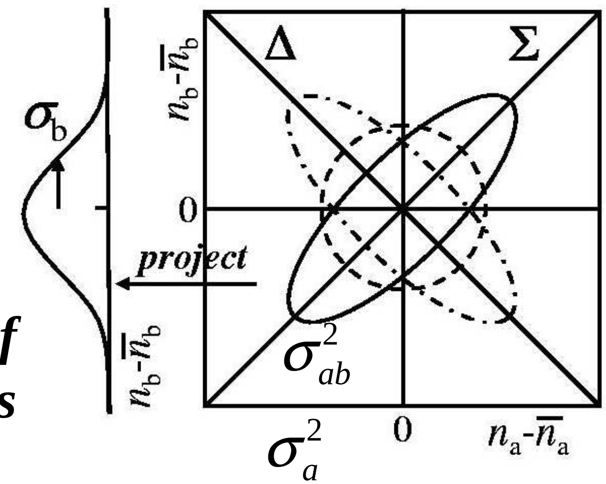
uniform local treatment of variances and covariances



bins *a* and *b*
single-particle space

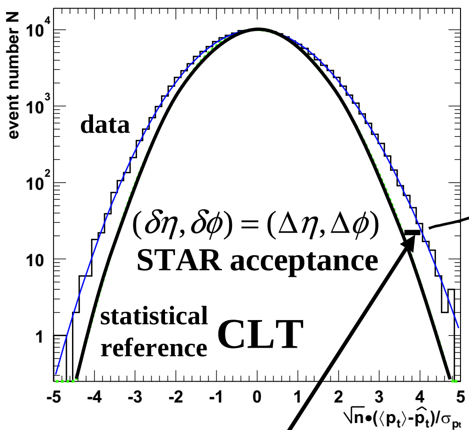


two-particle space

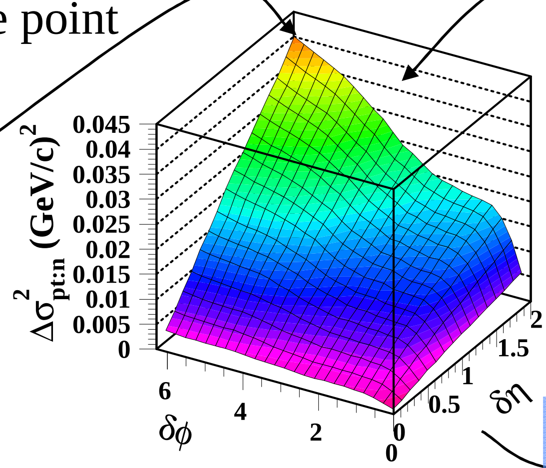


- Linearity: invariance under superposition of *uncorrelated* elements
- Locality: each pair of bins treated equivalently, based on Pearson’s coefficient

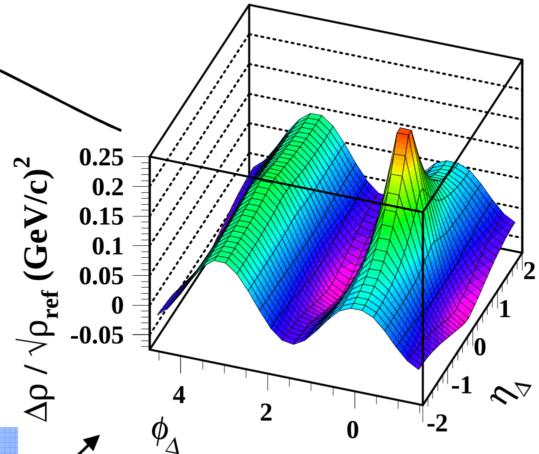
Correlations from Fluctuations



single point



2D scale integral



$\langle p_t \rangle$ fluctuation excess

scale dependence

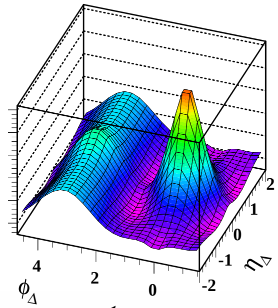
2D scale inversion

autocorrelation

$$\Delta\sigma^2_{p_t:n}\left(m\varepsilon_\eta, n\varepsilon_\phi\right) = 4\sum_{k=1}^{m}\sum_{l=1}^{n}\underbrace{\left(1-\frac{k-1/2}{m}\right)\left(1-\frac{l-1/2}{n}\right)}_{\text{kernel } K_{mn;kl}}\frac{\Delta A}{\sqrt{A_{ref}}}\left(k\varepsilon_\eta, l\varepsilon_\phi\right)$$

$$\Delta\sigma^2_{p_t:n}\left(\delta\eta,\delta\varphi\right) = 4\int_0^{\delta\eta} d\eta_\Delta \int_0^{\delta\varphi} d\varphi_\Delta K\left(\delta\eta,\delta\varphi;\eta_\Delta,\varphi_\Delta\right)\frac{\Delta\rho}{\sqrt{\rho_{ref}}}\left(\eta_\Delta,\varphi_\Delta\right)$$

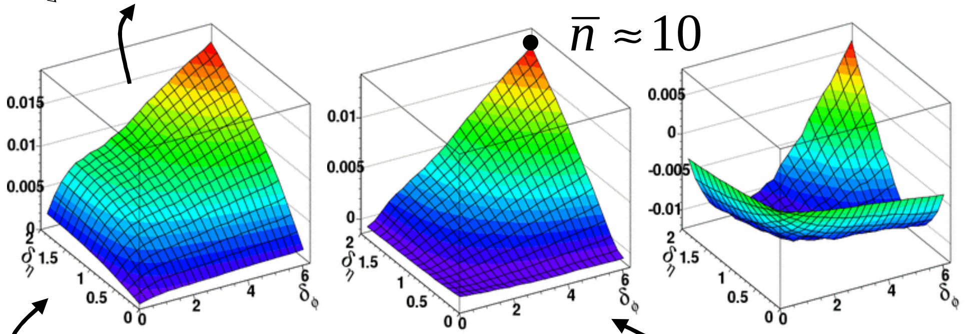
fluctuations $\Leftrightarrow$ ***integral equation*** $\Leftrightarrow$ correlations



Fluctuation Measure Design

peripheral Au-Au

$\bar{n} \approx 10$



$\bar{n} \approx 10$ at full acceptance!

$$\Delta\sigma^2_{p_t:n} \equiv \overline{(p_t - n\,\hat{p}_t)^2}/\bar{n} - \sigma^2_{\hat{p}_t}$$

$$\Delta\sigma^2_{p_t:n,old} \equiv \overline{(p_t - n\,\hat{p}_t)^2/n} - \sigma^2_{\hat{p}_t}$$

$$(\bar{n}-1)\left\langle \delta p_{ti}\delta p_{tj}\right\rangle_{i\neq j} \equiv$$

$$\overline{\frac{(p_t - n\,\hat{p}_t)^2}{n}\cdot\left\{\frac{\bar{n}-1}{n-1}\right\}} - \sigma^2_{\hat{p}_t}\overline{\left\{\frac{\bar{n}-1}{n-1}\right\}}$$

algebra of random variables vs algebra of ordinary variables

$$\overline{1/n} \neq 1/\bar{n}, \quad \overline{nm} \neq \bar{n}\,\bar{m}, \quad \overline{n^2} \neq \bar{n}^2$$

$$\bar{n}\Delta\sigma^2_{\langle p_t\rangle} \equiv \bar{n}\overline{\left(\langle p_t\rangle - \hat{p}_t\right)^2} - \sigma^2_{\hat{p}_t}$$

$$= \overline{\frac{(p_t - n\,\hat{p}_t)^2}{n}\cdot\frac{\bar{n}}{n}} - \sigma^2_{\hat{p}_t}$$

CLT $\neq$ 0!

$$\approx \sigma^2_{\hat{p}_t}\cdot\frac{\sigma_n^2}{2\bar{n}} \approx \sigma^2_{\hat{p}_t}/2 \neq 0$$

- Bias (sys. error) at small *n*
- Efficiencies, acceptances
- Superposition and linearity

Summary

- Basic random variables are n_+, n_-, p_t (p_{t+}, p_{t-})
- Physics lies on sum and difference variables
- Normalized variances $\leftrightarrow$ *per-particle*
- Statistical reference: central limit theorem
- Linearity and locality $\rightarrow$ optimum measures
- Scale dependence: fluctuations $\leftrightarrow$ correlations
- Algebra of random variables is not simple
- Small multiplicities challenge measure design