Relating Fluctuations and Correlations – PART I

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- I show how to express the mean-p_t fluctuation measure at full STAR acceptance scale as an integral of the two-particle number correlations on p_t x p_t space. This relates the measure defined in STAR paper nucl-ex/0308033, *“Event-wise <pt> fluctuations in Au+Au collisions at sqrt{s_NN} =130 GeV,”* to Eq.(1) in STAR paper nucl-ex/0408012, *“Two-particle correlations on transverse momentum and minijet dissipation in Au+Au collisions at sqrt{s_NN} =130 GeV.”*

- The general steps are sketched out first. For those interested in understanding the precise algebraic steps I present a pedagogical derivation in a following Appendix.

In STAR paper nucl-ex/0308033, “Event-wise <pt> fluctuations …” we introduced a new measure of non-statistical event-wise fluctuations in mean transverse momentum based on the difference between the total variance and that expected when there are no dynamical correlations:

$$\Delta\sigma^2_{p_t:n} \equiv \overline{n_j(\langle p_t \rangle_j - \hat{p}_t)^2} - \sigma^2_{\hat{p}_t}$$

(see Appendix – slide 7 for definition of symbols; overline denotes event-wise average)

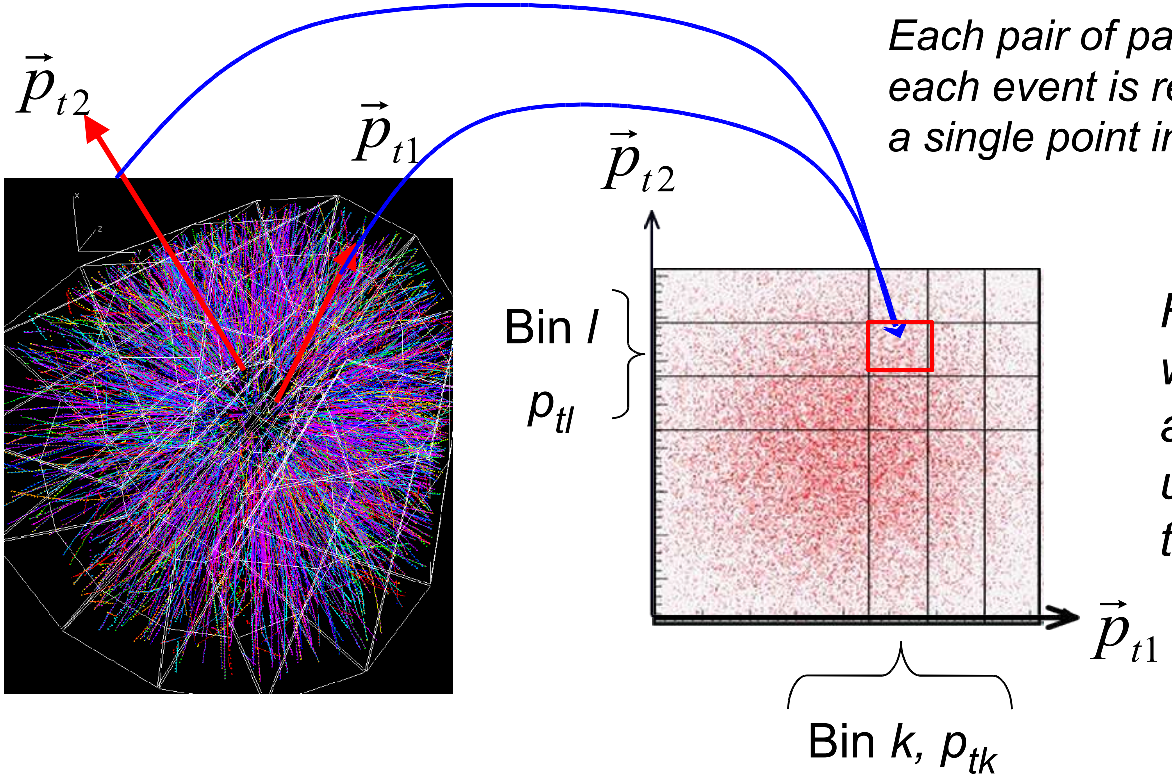
In order to relate this variance difference quantity to two-particle correlations we need to re-express $\Delta\sigma^2_{pt:n}$ in terms of <u>sums over pairs of particles</u>:

$$\Delta\sigma^2_{p_t:n} \cong \overline{\left\{ \frac{1}{n_j} \boxed{\sum_{i \neq i' = 1}^{n_j} (p_{t,ji} p_{t,ji'}} - \hat{p}_t^2) \right\}}$$

(see Appendix for complete algebraic details)

This is the first line in Eq.(1) in nucl-ex/0408012.

Next, relate the sum over real pairs of particles within each event (first term of preceding equation) to the two-particle number density:



Each pair of particles from each event is represented by a single point in the pt1 vs pt2 space.

Fill this binned space with all pairs of accepted particles using all events in the centrality bin.

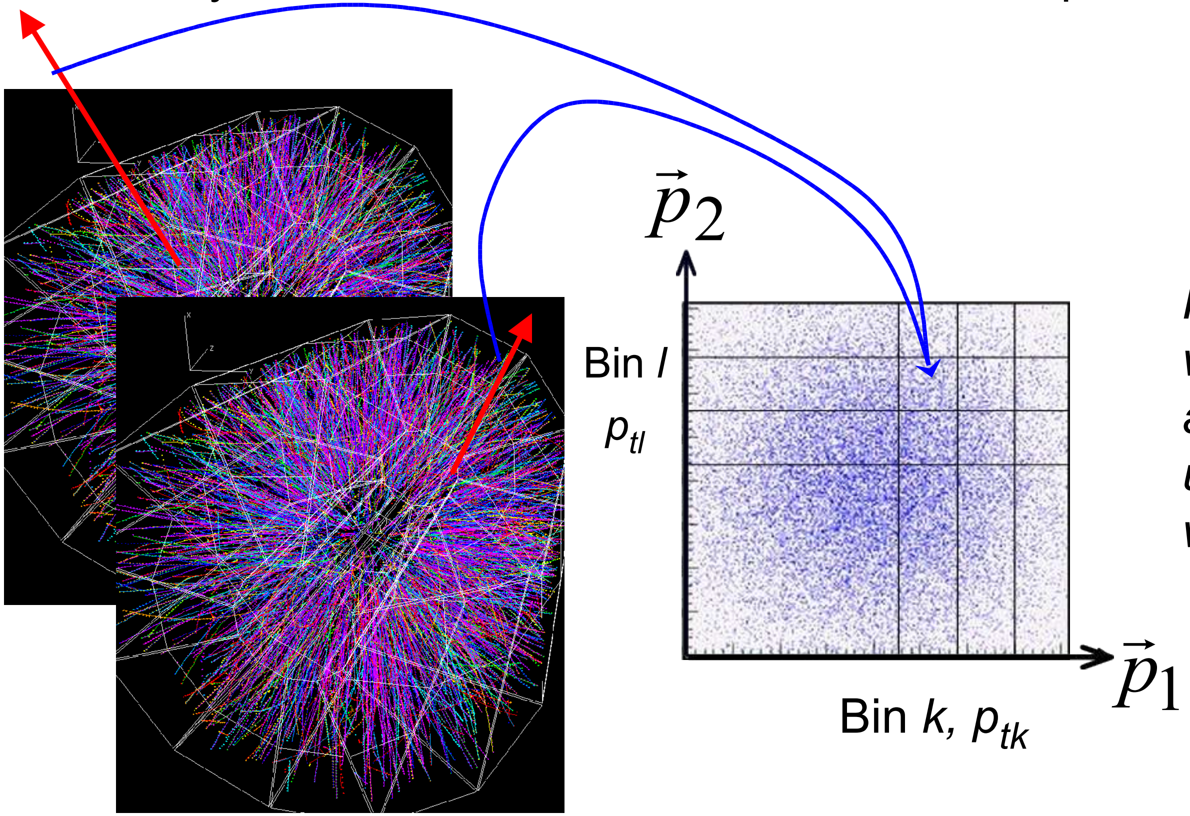
$$\overline{\sum_{i\neq i'=1} p_{t,ji}\,p_{t,ji'}} \cong \overline{\sum_{k,l} p_{t,k}\,p_{t,l}\,n^{sib}_{j,kl}} = \overline{\sum_{k,l} p_{t,k}\,p_{t,l}\,\varepsilon^{2}_{p_t}}\,\bar{\rho}_{sib}(p_{t,k},p_{t,l})$$

The sum of pairs in event *j*, averaged over events…

is approx. by a sum over bins, averaged over events, where n^{sib}=#real pairs in 2D bin *(k,l)*.

The latter, avg. number of sibling pairs in bin *(k,l)* is identified with sibling pair density times 2D bin area.

Similarly, relate the inclusive (mean-*pt*)^2 term to a sum over mixed-event pairs and the mixed pair density, which serves as the uncorrelated two-particle reference density:



Fill this binned space with all pairs of accepted particles using mixed-events within the centrality bin.

$$\hat{p}_t^{\,2} \equiv \left(\frac{1}{\bar{N}} \overline{\sum_{i=1} p_{t,ji}} \right)^2 \cong \frac{1}{\bar{N}^2} \overline{\overline{\sum_{k,l} p_{t,k} p_{t,l} n_{j,k} n_{j',l}}} = \frac{1}{\bar{N}^2} \sum_{k,l} p_{t,k} p_{t,l} \varepsilon_{p_t}^2 \bar{\rho}_{mix}(p_{t,k}, p_{t,l})$$

Mixed-event avg. (double overlines) of sum over 2D bins, where

$n_{j,k}$=#particles in bin *k*.

The latter, avg. number of mixed pairs in bin *(k,l)* is identified with mixed pair density times 2D bin area.

Combining these two parts gives the relationship between mean-*pt* fluctuation variance excess measure $\Delta\sigma^2_{pt:n}$ and two-particle correlations in Eq.(1) of nucl-ex/0408012 given by:

$$\begin{aligned}
\Delta\sigma^{2}_{p_t:n} &\cong \sum_{k,l} p_{t,k}\, p_{t,l}\, \varepsilon^{2}_{p_t} \left[\bar{\rho}_{sib}(p_{t,k}, p_{t,l}) - \bar{\rho}_{mix}(p_{t,k}, p_{t,l})\right] / \bar{N} \\
&\cong (1/\bar{N}) \iint_{(\Delta p_t)^2} dp_{t1}\, dp_{t2}\, p_{t1}\, p_{t2} \left[\bar{\rho}_{sib}(p_{t1}, p_{t2}) - \bar{\rho}_{mix}(p_{t1}, p_{t2})\right] \\
&= (1/\bar{N}) \iint_{(\Delta p_t)^2} dp_{t1}\, dp_{t2}\, p_{t1}\, p_{t2}\, \bar{\rho}_{mix}(p_{t1}, p_{t2}) \left[\hat{r}(p_{t1}, p_{t2}) - 1\right]
\end{aligned}$$

Note that we usually do not bin the two-particle densities directly in p_t, but rather use a mapping from p_t to $X(p_t)$ in order to achieve approx. uniform statistics in the bins. Also, in the future we plan to use transverse rapidity, y_t, which is another mapping, in order to optimally display the transverse string fragmentation dynamics, analogous to that in Lund string fragmentation models along the beam axis.

All the steps and details are given in the following Appendix.

Appendix

- Definition of symbols
- Manipulation of $\Delta\sigma^2_{pt:n}$ into sums over pairs of particles
- Derivation of lines 2 and 3 in Eq.(1) of nucl-ex/0408012.

Definition of Symbols:

ε = number of events in a centrality bin

j, j' = event indices

n_j = number of particles used in event j in acceptance

i, i' = particle indices

$\bar{N}$ = mean multiplicity for all events

$p_{t,ji}$ = transverse momentum magnitude of particle i in event j.

$\langle p_t \rangle_j$ = mean of transverse momentum magnitudes for all accepted particles in event j.

$\hat{p}_t$ = inclusive mean p_t for all accepted particles in all events

$\sigma^2_{\hat{p}_t}$ = inclusive p_t variance for all accepted particles in all events

Manipulation of $\Delta\sigma^2_{pt:n}$ into sums over pairs of particles

First, I show how the mean-p_t variance excess measure in Eq.(2) of STAR paper nucl-ex/0308033 can be manipulated into sums over pairs of particles from the same events (sibling pairs) and from mixed events (mixed pairs):

STAR variance excess measure

$$\Delta\sigma^2_{p_t:n} \equiv \frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon} n_j \left[\langle p_t \rangle_j - \hat{p}_t\right]^2 - \sigma^2_{\hat{p}_t}$$

[Eq.(2) in nucl-ex/0308033]

(expand symbols)

$$= \frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon} n_j \left[\frac{1}{n_j}\sum_{i=1}^{n_j} p_{t,ji} - \hat{p}_t\right]^2 - \frac{1}{\bar{N}\varepsilon}\sum_{j=1}^{\varepsilon}\sum_{i=1}^{n_j}\left(p_{t,ji} - \hat{p}_t\right)^2$$

(write out the squares)

$$= \frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon} n_j \left[\frac{1}{n_j^2}\sum_{i\neq i'=1}^{n_j} p_{t,ji}p_{t,ji'} + \frac{1}{n_j^2}\sum_{i=1}^{n_j} p_{t,ji}^2 - \frac{2\hat{p}_t}{n_j}\sum_{i=1}^{n_j} p_{t,ji} + \hat{p}_t^2\right]$$

$$- \frac{1}{\bar{N}\varepsilon}\sum_{j=1}^{\varepsilon}\sum_{i=1}^{n_j}\left(p_{t,ji}^2 - 2\hat{p}_t p_{t,ji} + \hat{p}_t^2\right)$$

(separate true pairs from "self" pairs)

$$\text{where} \quad \bar{N} \equiv \frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon} n_j, \quad \text{and} \quad \hat{p}_t \equiv \frac{1}{\bar{N}\varepsilon}\sum_{j=1}^{\varepsilon}\sum_{i=1}^{n_j} p_{t,ji}$$

continued,

(write out all the terms)

$$\Delta\sigma^2_{p_t:n} = \frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon}\frac{1}{n_j}\sum_{i\neq i'=1}^{n_j} p_{t,ji}p_{t,ji'} - \frac{2\hat{p}_t}{\varepsilon}\sum_{j=1}^{\varepsilon} n_j \frac{1}{n_j}\sum_{i=1}^{n_j} p_{t,ji}$$

$$+\frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon} n_j \hat{p}_t^2 + \frac{2\hat{p}_t}{\bar{N}\varepsilon}\sum_{j=1}^{\varepsilon}\sum_{i=1}^{n_j} p_{t,ji} - \frac{1}{\bar{N}\varepsilon}\sum_{j=1}^{\varepsilon}\sum_{i=1}^{n_j}\hat{p}_t^2$$

$$+\frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon}\frac{1}{n_j}\sum_{i=1}^{n_j} p^2_{t,ji} - \frac{1}{\bar{N}\varepsilon}\sum_{j=1}^{\varepsilon}\sum_{i=1}^{n_j} p^2_{t,ji}$$

(then collect them)

$$= \frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon}\frac{1}{n_j}\sum_{i\neq i'=1}^{n_j} p_{t,ji}p_{t,ji'} + (-2\bar{N} + \bar{N} + 2 - 1)\hat{p}_t^2$$

$$+\frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon}\left(1-\frac{n_j}{\bar{N}}\right)\underbrace{\frac{1}{n_j}\sum_{i=1}^{n_j} p^2_{t,ji}}_{\equiv \left\langle p_t^2\right\rangle_j}$$

(define)

continued,

$$\Delta\sigma^2_{p_t:n} = \frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon}\frac{1}{n_j}\sum_{i\neq i'=1}^{n_j} p_{t,ji}p_{t,ji'} - (\bar{N}-1)\hat{p}_t^2 + \underbrace{\frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon}\left(1-\frac{n_j}{\bar{N}}\right)\left\langle p_t^2\right\rangle_j}$$

This term vanishes exactly if mean-p_t^2 is not correlated with n_j. For STAR applications this term is small compared to differences between the first two terms and will be neglected.

$$\Delta\sigma^2_{p_t:n} \cong \frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon}\frac{1}{n_j}\sum_{i\neq i'=1}^{n_j} p_{t,ji}p_{t,ji'} - (\bar{N}-1)\hat{p}_t^2$$

$$\boxed{\Delta\sigma^2_{p_t:n} \cong \frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon}\left\{\frac{1}{n_j}\sum_{i\neq i'=1}^{n_j}\left(p_{t,ji}p_{t,ji'} - \hat{p}_t^2\right)\right\}}$$

This is the first line in Eq.(1) in nucl-ex/0408012

Derivation of lines 2 and 3 in Eq.(1) of nucl-ex/0408012.

In Tutorials 2 and 3 I introduced the normalized pair density ratio which can be related to the two-particle correlation. Starting with this measured ratio in two-dimensional $p_t \times p_t$ space I will explain how the correlation density and the combination of sums over sibling and mixed pairs on the preceding pages can be related.

Normalized ratio of sibling-to-mixed particle pair densities

$$\hat{r}_{kl} = \underbrace{\frac{\sum_{j=1}^{\varepsilon} n^{sib}_{j,kl}}{\sum_{j=1}^{\varepsilon} n_j (n_j - 1)}}_{\text{Fraction of total sibling pairs in bin } (k,l)} \div \underbrace{\frac{\sum_{j \neq j'} n_{j,k} n_{j',l}}{\sum_{j \neq j'} n_j n_{j'}}}_{\text{Fraction of total mixed pairs in bin } (k,l)} \left. \vphantom{\frac{\sum}{\sum}} \right\}$$

$n^{sib}_{j,kl}$ = number of sibling pairs in 2D $p_t \otimes p_t$ bin (k,l) in event j

$n_{j,k}$ = number of particles in p_t bin k, in event j

Represent event averaged, bin-wise sibling pair fraction with integral over 2D sibling pair density, ρ_{sib}.

$$\left\{ \frac{\int_{\varepsilon_k} \int_{\varepsilon_l} dp_{t1} dp_{t2} \rho_{sib}(p_{t1}, p_{t2})}{\iint_{(\Delta p_t)^2} dp_{t1} dp_{t2} \rho_{sib}(p_{t1}, p_{t2})} \cong \frac{\sum_{j=1}^{\varepsilon} n^{sib}_{j,kl}}{\sum_{j=1}^{\varepsilon} n_j (n_j - 1)} \right.$$

where ε_k, Δp_t are the p_t bin size and acceptance

Define the bin-wise average density; assume uniform bin sizes ε_{pt}, and introduce bin momentum $p_{t,k}$:

$$\int_{\varepsilon_k}\int_{\varepsilon_l} dp_{t1}dp_{t2}\,\rho_{sib}(p_{t1},p_{t2}) \equiv \varepsilon_k\varepsilon_l\,\bar{\rho}_{sib}(p_{t,k},p_{t,l}) = \varepsilon_{p_t}^2\,\bar{\rho}_{sib}(p_{t,k},p_{t,l})$$

Similarly, represent the event averaged, bin-wise mixed pair fraction as integral over ρ_{mix} and define bin-wise average.

$$\frac{\int_{\varepsilon_k}\int_{\varepsilon_l} dp_{t1}dp_{t2}\,\rho_{mix}(p_{t1},p_{t2})}{\iint_{(\Delta p_t)^2} dp_{t1}dp_{t2}\,\rho_{mix}(p_{t1},p_{t2})} \cong \frac{\sum_{j\neq j'} n_{j,k}n_{j',l}}{\sum_{j\neq j'} n_j n_{j'}}$$

$$\int_{\varepsilon_k}\int_{\varepsilon_l} dp_{t1}dp_{t2}\,\rho_{mix}(p_{t1},p_{t2}) \equiv \varepsilon_k\varepsilon_l\,\bar{\rho}_{mix}(p_{t,k},p_{t,l}) = \varepsilon_{p_t}^2\,\bar{\rho}_{mix}(p_{t,k},p_{t,l})$$

Normalize the sibling and mixed-event densities to the event-averaged number of pairs; express histogram ratio in terms of densities:

$$\iint_{(\Delta p_t)^2} dp_{t1}dp_{t2}\,\rho_{sib/mix}(p_{t1},p_{t2}) \equiv \overline{N(N-1)}$$

$$\hat{r}_{kl} = \frac{\varepsilon_{p_t}^2\,\bar{\rho}_{sib}(p_{t,k},p_{t,l})}{\varepsilon_{p_t}^2\,\bar{\rho}_{mix}(p_{t,k},p_{t,l})} = \frac{\bar{\rho}_{sib}(p_{t,k},p_{t,l})}{\bar{\rho}_{mix}(p_{t,k},p_{t,l})}$$

Using the results on page 11 and replacing the sums over pairs with sums over 2D p_t x p_t bins we get,

$$\Delta\sigma^2_{p_t:n} = \frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon}\frac{1}{n_j}\sum_{k,l} p_{t,k}\,p_{t,l}\,n^{sib}_{j,kl} - \frac{(\overline{N}-1)}{\varepsilon^2\,\overline{N}^2}\sum_{j,j'=1}^{\varepsilon}\sum_{k,l} p_{t,k}\,p_{t,l}\,n_{j,k}\,n_{j',l}$$

(factor out $p_{tk}p_{tl}$)

$$= \sum_{k,l} p_{t,k}\,p_{t,l}\left[\frac{1}{\varepsilon}\sum_{j=1}^{\varepsilon}\frac{n^{sib}_{j,kl}}{n_j} - \underbrace{\frac{(\overline{N}-1)}{\varepsilon^2\,\overline{N}^2}\sum_{j\neq j'=1}^{\varepsilon} n_{j,k}\,n_{j',l} - \frac{(\overline{N}-1)}{\varepsilon^2\,\overline{N}^2}\sum_{j=1}^{\varepsilon} n_{j,k}\,n_{j,l}}_{\text{(separate sums over different and same events)}}\right]$$

If we assume $n_j = \overline{N}$, then

$$\begin{aligned}
\Delta\sigma^2_{p_t:n} &\cong \sum_{k,l} p_{t,k}\,p_{t,l}\left[\frac{\varepsilon^2_{p_t}}{\overline{N}}\,\overline{\rho}_{sib}(p_{t,k},p_{t,l}) - \frac{\varepsilon^2_{p_t}}{\overline{N}}\,\overline{\rho}_{mix}(p_{t,k},p_{t,l})\right] \\
&\quad + \sum_{k,l} p_{t,k}\,p_{t,l}\,\frac{\varepsilon^2_{p_t}}{\varepsilon\overline{N}}\left(\overline{\rho}_{mix}(p_{t,k},p_{t,l}) - \frac{\overline{N}-1}{\overline{N}}\,\overline{\rho}_{sib}(p_{t,k},p_{t,l})\right) \\
&\cong \sum_{k,l} p_{t,k}\,p_{t,l}\,\varepsilon^2_{p_t}\left[\overline{\rho}_{sib}(p_{t,k},p_{t,l}) - \overline{\rho}_{mix}(p_{t,k},p_{t,l})\right]/\,\overline{N}
\end{aligned}$$

This term is of order 1/ε relative to the leading term; neglect for large event samples.

$$\Delta\sigma^2_{p_t:n} \cong (1/\overline{N})\underbrace{\iint_{(\Delta p_t)^2}}_{\text{In the limit of very small } p_t \text{ bins}} dp_{t1}\,dp_{t2}\,p_{t1}\,p_{t2}\left[\overline{\rho}_{sib}(p_{t1},p_{t2}) - \overline{\rho}_{mix}(p_{t1},p_{t2})\right]$$

This is the second line in Eq.(1) in nucl-ex/0408012

Using the definition of histogram ratio *r,* evaluating (mean p_t)^2 using the mixed density, and using the density normalization definition, the final expression is written compactly as follows:

$$\Delta\sigma^2_{p_t:n} \cong (1/\bar{N}) \iint_{(\Delta p_t)^2} dp_{t1}dp_{t2}\,p_{t1}p_{t2}\,\bar{\rho}_{mix}(p_{t1},p_{t2})\left[\hat{r}(p_{t1},p_{t2})-1\right]$$

$$\hat{p}_t^2 = \frac{\iint_{(\Delta p_t)^2} dp_{t1}dp_{t2}\,p_{t1}p_{t2}\,\bar{\rho}_{mix}(p_{t1},p_{t2})}{\iint_{(\Delta p_t)^2} dp_{t1}dp_{t2}\,\bar{\rho}_{mix}(p_{t1},p_{t2})} = \frac{\iint_{(\Delta p_t)^2} dp_{t1}dp_{t2}\,p_{t1}p_{t2}\,\bar{\rho}_{mix}(p_{t1},p_{t2})}{\overline{N(N-1)}}$$

$$\Delta\sigma^2_{p_t:n} \cong (\bar{N}-1)\,\hat{p}_t^2 \left\langle \hat{r}(p_{t1},p_{t2})-1\right\rangle,$$

This is the third line in Eq.(1) in nucl-ex/0408012

where the bracket represents a weighted average given in general by:

$$\left\langle \Re \right\rangle \equiv \frac{\iint_{(\Delta p_t)^2} dp_{t1}dp_{t2}\,p_{t1}p_{t2}\,\bar{\rho}_{mix}(p_{t1},p_{t2})\Re}{\iint_{(\Delta p_t)^2} dp_{t1}dp_{t2}\,p_{t1}p_{t2}\,\bar{\rho}_{mix}(p_{t1},p_{t2})}$$

The last two equations summarize the integral relation between correlations and fluctuation measures used in nucl-ex/0408012 for full acceptance scale. <u>Similar derivations can be applied to any other binned quantity.</u>