A   Helix Parametrization

The trajectory of a charged particle in a static uniform magnetic field with $\vec{B} = (0, 0, B_z)$ is a helix. In principle five [8] parameters are needed to define such a helix. From the various possible parametrizations we describe here the version which is well suited for the geometry of a collider experiment and therefore used for the implementation of the StHelix class.

This parametrization describes the helix in Cartesian coordinates, where x, y and z are expressed as functions of the track length s.

$$x(s) \;=\; x_0 + \frac{1}{\kappa}[\cos(\Phi_0 + h\, s\, \kappa\, \cos\lambda) - \cos\Phi_0] \tag{1}$$

$$y(s) \;=\; y_0 + \frac{1}{\kappa}[\sin(\Phi_0 + h\, s\, \kappa\, \cos\lambda) - \sin\Phi_0] \tag{2}$$

$$z(s) \;=\; z_0 + s\, \sin\lambda \tag{3}$$

where here and in the following:

s is the path length along the helix

x_0, y_0, z_0 is the starting point at $s = s_0 = 0$

λ is the dip angle

κ is the curvature, i.e. $\kappa = 1/R$

B is the z component of the homogeneous magnetic field ($B = (0, 0, B_z)$)

q is charge of the particle in units of positron charge

h is the sense of rotation of the projected helix in the xy-plane,
      i.e. $h = -\text{sign}(qB) = \pm 1$

Φ_0 is the azimuth angle of the starting point (in cylindrical coordinates) with respect to the helix axis
      ($\Phi_0 = \Psi - h\pi/2$)

Ψ is the $\arctan(dy/dx)_{s=0}$, i.e. the azimuthal angle of the track direction at the starting point.

The meaning of the different parameters is visualized in Fig. A.1.

A.1   Calculation of the particle momentum

The circle fit in the xy-plane gives the center of the fitted circle (x_c, y_c) and the curvature $\kappa = 1/R$ while the linear fit gives z_0 and $\tan\lambda$. The phase of the helix (see Fig. A.1) is defined as follows:

$$\Phi_0 = \arctan\left(\frac{y_0 - y_c}{x_0 - x_c}\right) \tag{4}$$

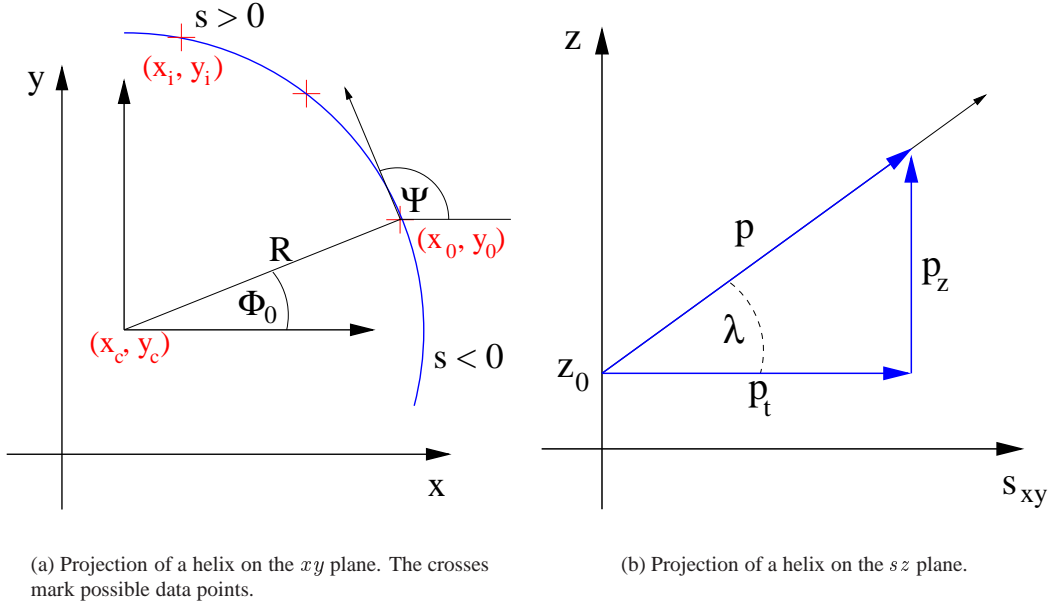[8]see A.8 for a detailed discussion on the number of parameters needed.

(a) Projection of a helix on the xy plane. The crosses mark possible data points.

(b) Projection of a helix on the sz plane.

Figure A.1: Helix parametrization

The reference point (x_0, y_0) is then calculated as follows:

$$
\begin{aligned}
x_0 &= x_c + \frac{\cos \Phi_0}{\kappa} & (5) \\
y_0 &= y_c + \frac{\sin \Phi_0}{\kappa} & (6)
\end{aligned}
$$

and the helix parameters can be evaluated as:

$$
\begin{aligned}
\Psi &= \Phi_0 + h\pi/2 & (7) \\
p_\perp &= c\, q\, B/\kappa & (8) \\
p_z &= p_\perp \tan \lambda & (9) \\
p &= \sqrt{p_\perp^2 + p_z^2} & (10)
\end{aligned}
$$

where κ is the curvature in $[\text{m}^{-1}]$, B the value of the magnetic field in [Tesla], c the speed of light in [m/ns] (≈ 0.3) and $p_\perp$ and p_z are the transverse and longitudinal momentum in [GeV/c].

A.2   Distant measure

The minimal squared distance M_i between a helix and a point i with position (x_i, y_i, z_i) is given by

$$M_i = M_i^{(xy)} + M_i^{(z)} \tag{11}$$
$$M_i = (x_i - x(s'))^2 + (y_i - y(s'))^2 + (z_i - z(s'))^2 \tag{12}$$
$$\tag{13}$$

In literature one finds the following approach to solve this problem analytically by neglecting $M_i^{(z)}$ in the derivatives.

$$\frac{dM_i^{xy}}{ds} = 0 \tag{14}$$

This formula can only serve to derive an approximation for the real distance. For large dip angles the errors become large depending also on the actual helix parameters. The advantage is that s' can be calculated analytically:

$$s' = \frac{1}{h\kappa \cos\lambda} \arctan\left(\frac{(y_i - y_0)\cos\Phi_0 - (x_i - x_0)\sin\Phi_0}{1/\kappa + (x_i - x_0)\cos\Phi_0 + (y_i - y_0)\sin\Phi_0}\right) \tag{15}$$

Note, that this formula can **not** be used to derive the distance of closest approach to a point. In order to derive the distance of closest approach the following equation has to be solved:

$$\frac{dM_i}{ds} = 0 \tag{16}$$

which can be written as

$$2\left(x_i - x_0 - \frac{\cos(\Phi_0 + hs\kappa \cos\lambda) - \cos\Phi_0}{\kappa}\right)\sin(\Phi_0 + hs\kappa \cos\lambda)\, h\cos\lambda -$$
$$2\left(y_i - y_0 - \frac{\sin(\Phi_0 + hs\kappa \cos\lambda) - \sin\Phi_0}{\kappa}\right)\cos(\Phi_0 + hs\kappa \cos\lambda)\, h\cos\lambda -$$
$$2\left(z_i - z_0 - s\sin\lambda\right)\sin\lambda = 0 \tag{17}$$

The root of eq. 17 can easily be found with the Newton or *regula falsi* method with s' from eq. 15 as starting value. For the Newton method the second derivative is needed as well.

$$\frac{d^2 M_i}{ds^2} = 0 \tag{18}$$

which is

$$
\begin{aligned}
2 \quad & \left(\sin(\Phi_0 + hs\kappa\cos\lambda)\right)^2 h^2\cos^2\lambda + \\
2 \quad & \left(x_i - x_0 - \frac{\cos(\Phi_0 + hs\kappa\cos\lambda) - \cos\Phi_0}{\kappa}\right) \\
& \cos(\Phi_0 + hs\kappa\cos\lambda)h^2\kappa\cos^2\lambda + \\
2 \quad & \left(\cos(\Phi_0 + hs\kappa\cos\lambda)\right)^2 h^2\cos^2\lambda + \\
2 \quad & \left(y_i - y_0 - \frac{\sin(\Phi_0 + hs\kappa\cos\lambda) - \sin\Phi_0}{\kappa}\right) \\
& \sin(\Phi_0 + hs\kappa\cos\lambda)h^2\kappa\cos^2\lambda + \\
2 \quad & \sin^2\lambda = 0
\end{aligned}
\tag{19}
$$

A.3   Distance of closest approach between two helices

The closest distance between two helices H_1 and H_2 is a problem which again can be solved analytically only in 2 dimensions, i.e., in the xy-plane. The solution in 3 dimensions cannot even be solved by standard numerical methods (as the Newton method) but requires more sophisticated method since we have to find 2 unknown parameters s_1 and s_2 in

$$
\frac{\mathrm{d}^2 M\left(s_1, s_2\right)}{\mathrm{d}s_1 \mathrm{d}s_2} = 0
\tag{20}
$$

where M is the distance between the two helices at s_1 and s_2.
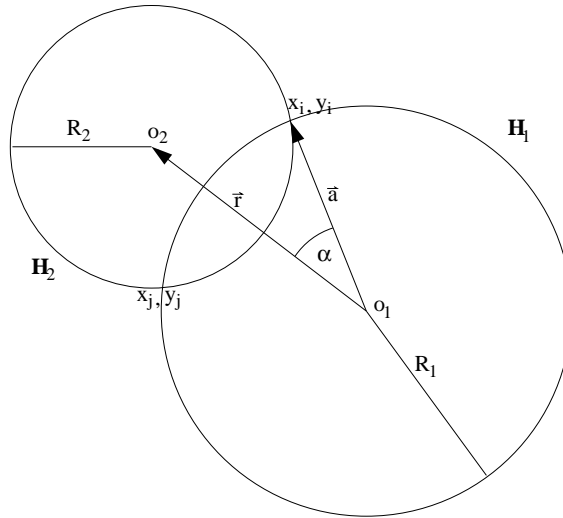
Figure A.2: Two intersecting helices in the xy-plane

In the xy-plane:
Given two helices with radii R_1 and R_2 and centers in the xy-plane $o_1 = (x_{c1}, y_{c1})$ and $o_2 = (x_{c2}, y_{c2})$ we have to find vector $\vec{a}$ as depicted in Fig. A.3. The angle α can be calculated as:

$$\cos\alpha = \frac{R_1^2 + |\vec{r}|^2 - R_2^2}{2R_1|\vec{r}|} \tag{21}$$

where $\vec{r}$ is the vector between the two centers. The absolute coordinates of one intersection point (measured from o_1) can be obtained by calculating vector $\vec{a}$ and adding o_1.

$$x_i = x_{c1} + R_1[(x_{c2} - x_{c1})\cos\alpha - (y_{c2} - y_{c1})\sin\alpha]/|\vec{r}|; \tag{22}$$

$$y_i = y_{c1} + R_1[(x_{c2} - x_{c1})\sin\alpha + (y_{c2} - y_{c1})\cos\alpha]/|\vec{r}|; \tag{23}$$

If $\cos\alpha$ is exactly 1 we have only one solution. For the case $\cos\alpha < 1$ we get two valid intersection points (x_i, y_i) and (x_j, y_j) where the latter is simply given by:

$$x_j = x_{c1} + R_1[(x_{c2} - x_{c1})\cos\alpha + (y_{c2} - y_{c1})\sin\alpha]/|\vec{r}|; \tag{24}$$

$$y_j = y_{c1} + R_1[(y_{c2} - y_{c1})\cos\alpha - (x_{c2} - x_{c1})\sin\alpha]/|\vec{r}|; \tag{25}$$

In the case $\cos\alpha > 1$ the circles do not intersect. Then the distance of closest approach is simply given by the intersection of a line between the two centers and the two helices. For helix H_1 we get:

$$x = x_{c1} + R_1(x_{c2} - x_{c1})/|\vec{r}|; \tag{26}$$

$$y = y_{c1} + R_1(y_{c2} - y_{c1})/|\vec{r}|; \tag{27}$$

In 3 dimensions:
Usually an iteration method is applied which uses the intersection points in the xy-plane as start values. Care has to be taken if both helices have different dip angle λ since the start values then significantly deviate from the actual solution.

A.4   Intersection with a cylinder (ρ=const)

In order to obtain the path length s at which the helix intersects with a cylinder of given radius ρ we have to solve the following equation:

$$\rho^2 = x(s)^2 + y(s)^2 \tag{28}$$

Using eq. 1 and 2 we obtain the two analytic solutions for s_1 and s_2:

$$\begin{aligned}
s_{1/2} = &-\Big(\Phi_0 + 2\arctan\Big[\big(2\,y_0\,\kappa - 2\,\sin\Phi_0 \pm \big[-\kappa^2\left(-4\,\rho^2 + 4\,y_0^2 - 2\,\rho^2\kappa^2 x_0^2 - \\
&2\,\rho^2\kappa^2 y_0^2 + 2\,x_0^2\kappa^2 y_0^2 + \rho^4\kappa^2 + x_0^4\kappa^2 + y_0^4\kappa^2 - 4\,x_0^3\kappa\,\cos\Phi_0 + \\
&4\,x_0^2\cos^2\Phi_0 - 4\,y_0^2\cos^2\Phi_0 - \\
&4\,y_0^3\kappa\,\sin\Phi_0 + 4\,\rho^2\kappa\,x_0\,\cos\Phi_0 + 4\,\rho^2\kappa\,y_0\,\sin\Phi_0 - 4\,x_0^2\kappa\,y_0\,\sin\Phi_0 - \\
&4\,y_0^2\kappa\,x_0\,\cos\Phi_0 + 8\,x_0\,\cos\Phi_0\,y_0\,\sin\Phi_0\big)\big]^{1/2}\Big) / \\
&\big(-\rho^2\kappa^2 + 2 + x_0^2\kappa^2 + 2\,\cos\Phi_0 + y_0^2\kappa^2 - \\
&2\,x_0\,\kappa - 2\,x_0\,\kappa\,\cos\Phi_0 - 2\,y_0\,\kappa\,\sin\Phi_0\big)\Big]\big)\,h^{-1}\kappa^{-1}\,(\cos\lambda)^{-1}
\end{aligned} \tag{29}$$
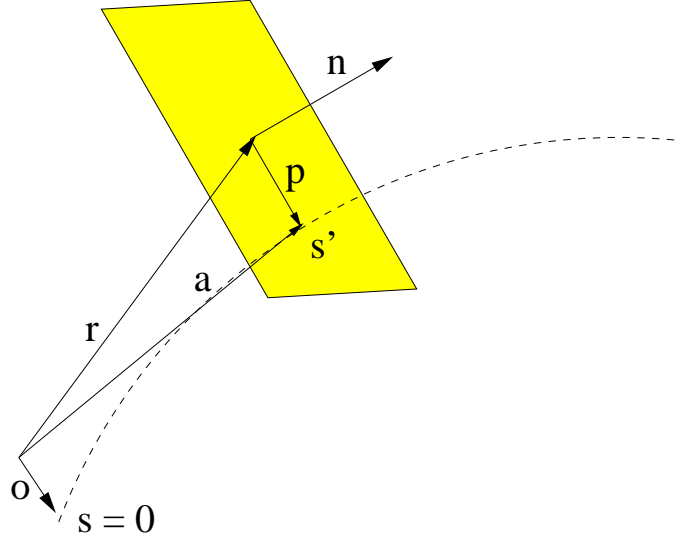
A.5   Intersection with a plane

Figure A.3: Intersection of a helix with a plane

Any plane can be described by its normal vector $\vec{n}$ (orientation) and an arbitrary point in this plane $\vec{r}$ (position). The vector $\vec{p}$ which describes the intersection point must fulfil:

$$\vec{p} \cdot \vec{n} = 0. \tag{30}$$

Hence:

$$(\vec{a} - \vec{r}) \cdot \vec{n} = 0. \tag{31}$$

where $\vec{a}$ is given by $\vec{a} = (x(s'), y(s'), z(s'))$ as described in eq. 1–3. In order to obtain the path length s' where the helix intersects with the plane the following equation has to be solved:

$$x(s)n_x + y(s)n_y + z(s)n_z - \vec{r} \cdot \vec{n} \quad = \tag{32}$$
$$A + n_x \cos S + n_y \sin S + \kappa n_z s \sin \lambda \quad = \quad 0 \tag{33}$$

where:

$$A \quad = \quad \kappa(\vec{o} \cdot \vec{n} - \vec{r} \cdot \vec{n}) - n_x \cos \Phi_0 - n_y \sin \Phi_0 \tag{34}$$
$$S \quad = \quad hs\kappa \cos \lambda + \Phi_0 \tag{35}$$

The root of eq. 33 can now easily be determined by a suitable numerical method (Newton).

A.6   Limitations

The only non-numerical limitations of this parametrization are:

$$- \pi/2 < \lambda \quad < \quad \pi/2 \tag{36}$$

$$\kappa \quad > \quad 0 \tag{37}$$

A.7   Case $B = 0$

For the special case $B = 0$ the trajectory becomes a straight line, i.e. $\kappa = 0$ and $R = \infty$. Care must be taken in the numerical calculation of the parametrization because of the singularity in eq. 1 and 2. The correct form is:

$$x(s) \quad = \quad x_0 - sh \cos \lambda \sin \Phi_0 \tag{38}$$

$$y(s) \quad = \quad y_0 + sh \cos \lambda \cos \Phi_0 \tag{39}$$

$$z(s) \quad = \quad z_0 + s \, \sin \lambda \tag{40}$$

Important: For $B = 0$ the sense of rotation is ill defined. All what matters is that $\Phi_0 = \Psi - h\pi/2$ is done correctly, i.e. with the same arbitrary h. In the following we assume $h = +1$ for convenience.

Eq. 15 then reads as:

$$s' = \frac{1}{\cos \lambda} \left[(y_i - y_0) \cos \Phi_0 - (x_i - x_0) \sin \Phi_0 \right] \tag{41}$$

Eq. 17 can now be solved analytically;

$$\frac{\mathrm{d} M_i^{dca}}{\mathrm{d} s} = 0 \tag{42}$$

gives:

$$s^{dca} \quad = \quad \cos \lambda \cos \Phi_0 (y_i - y_0) -$$
$$\cos \lambda \sin \Phi_0 (x_i - x_0) +$$
$$\sin \lambda (z_i - z_0) \tag{43}$$

The solution for the intersection with a cylinder (eq. 29) now reads:

$$s_{1/2} \quad = \quad [x_0 \cos \lambda \, \sin \Phi_0 - y_0 \, \cos \lambda \cos \Phi_0 \pm \tag{44}$$

$$\left[-\cos^2 \lambda \, \left(2 \, x_0 \, \cos \Phi_0 \, y_0 \, \sin \Phi_0 - \rho^2 + \right. \right. \tag{45}$$

$$\left. \left. y_0^2 - y_0^2 \, \cos^2 \Phi_0 + x_0^2 \cos^2 \Phi_0 \right) \right]^{1/2} \right] \cos^2 \lambda \tag{46}$$

The same holds for the intersection of a helix with a plane where in case of zero curvature eq. 33 can be solved analytically.

$$s' = \frac{\vec{r} \cdot \vec{n} - \vec{o} \cdot \vec{n}}{-n_x \cos \lambda \sin \Phi_0 + n_y \cos \lambda \cos \Phi_0 + n_z \sin \lambda} \tag{47}$$

A.8   Why are there only 5 independent helix parameters?

Imagine an arbitrary helix sitting in 3D space. What is required to completely specify it ?

1. the line coinciding with the axis - if its oriented in any arbitrary direction, then this requires 4 parameters, or 2 direction angles (theta and phi in usual spherical coordinates) plus 2 more coordinates to locate the line in a plane perpendicular to this direction.

   For the special case in STAR we always fix the direction parallel to the z-axis so this reduces the number of this subset of parameters from 4 to 2.

   So these are the (x,y) coordinates of the center of the circular projection onto the x-y plane.

2. Then we must give the radius of the circular projection - 1 more param,

3. Next we must specify the pitch and the handedness. - 2 more params.

4. Finally, we have to give a phase angle or some single number that tells us where this thing is actually sitting w.r.t. some given plane. For STAR this could be the phase angle at the point where the helix intersects the x-y plane. - 1 more parameter.

So, in general there are 7 continuously varying parameters plus a handedness switch. For the special case STAR uses there are then 5 independent parameters plus the left handed/right handed switch. So yes, 6 parameters are required. But for track fitting purposes only 5 are relevant. The handedness of the particle's trajectory will be determined by the sign of $B_z \times$ charge (using the actual charge) and the sign of the z-component of momentum (using the actual p_z momentum value). However, in general the charge sign and p_z direction are not known, based on track fitting alone. These signs must be assumed using some selection criteria, usually that the particle is moving "outward and away" from the general area of the beam. These two signs are not independent of each other but must be chosen to give a path consistent with the handedness that the space point positions require. So there is one algebraic sign that is ambiguous and that we have to choose. Having done this there remain 5 independent fitting parameters. In our track parametrization we put the choice of algebraic sign into the both that of the charge and the tanl parameter, consistently we hope.

So to summarize, there are 6 parameters, one is a sign selected by some criteria, the remaining 5 are varied to fit the space points. The track parameter error matrix is then 5×5 symmetric, and thus includes 15 distinct quantities.

Text from Lanny Ray written during an email exchange on this very topic.