A PEDESTRIAN'S GUIDE TO INTERFEROMETRY

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1 Introduction

In the collision of two heavy ions at RHIC or LHC energies, it is thought that a quark-gluon plasma will be formed. The defining characteristic of such a state of matter is a deconfinement region significantly larger than the ~ 1 fm radii of the individual hadrons. The only *direct* method of probing the spatial distribution of hadronic collisions is to measure correlations between pairs of emitted particles. This approach, especially when applied to pairs of identical particles, is usually referred to as Hanbury-Brown–Twiss (HBT) interferometry, in recognition of the pioneering work of Hanbury-Brown and Twiss[1] in applying this technique to optical astronomy.

Given the crucial importance of such information in understanding the space-time development of nucleus-nucleus collisions, it behooves us not to trivialize our interpretation of such measurements. Like any powerful tool, interferometry should be used with some caution– the passage from the observed phenomena to the desired information requires some care. Only a naif would assume that this delicacy is somehow unique to HBT. The lesson of the first five years of truly relativistic nuclear collisions has been that all simplistic analyses of the data, such as transverse momenta spectra[2] or J/Ψ production and suppression[3], are much more subtle than initially advertised.

The material presented here is designed to provide a self-contained introduction to the most important issues confronting HBT measurements. It is not meant as a review of entire field; those interested should consult available references[4, 5, 6]. Nor should students regard it as a substitute for reading the original literature– this is always a worthwhile endeavor. What I do hope to achieve here is a tutorial document that will expedite the difficult passage from a novice to a practicing interferometer.

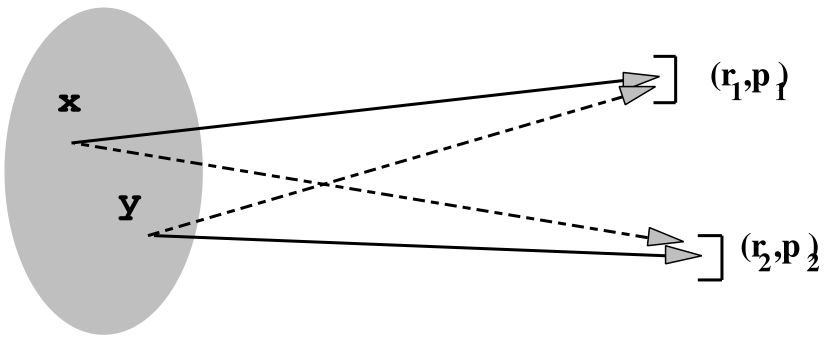

Figure 1: The two indistinguishable processes leading to the detection of identical pions with momentum p_1 at r_1 and p_2 at r_2.

2 Correlation Basics

This section presents the "usual" derivation of the HBT effect. While later sections will demonstrate that this is at best a motivation rather than a derivation, it will suffice to define some of the relevant variables, as well as demonstrating problems in interpretation that can arise even at this elementary level. In fact, nearly all of the issues that affect the interpretation of current experiments may be discussed in this context; readers who prefer the more advanced treatment of Sections 3 and 4 should not lose sight of the subtleties that are present even at an elementary level.

2.1 First Derivation of the Correlation Function

Assume two identical bosons (typically pions) are produced with momentum $\vec{p}_1$ and $\vec{p}_2$ by a source $\rho(r)$ extended in space and time. Let a typical pair of production points be labeled x and y, and denote the positions of detecting the pion with p_i as r_i. If we further assume that the pion wave-functions may be described by plane-waves, the amplitude for this event may be written as

$$A_{12} = \frac{1}{\sqrt{2}}\left[e^{ip_1\cdot(r_1-x)}e^{ip_2\cdot(r_2-y)} + e^{ip_1\cdot(r_1-y)}e^{ip_2\cdot(r_2-x)}\right]. \tag{1}$$

The "second" term is required to symmetrize the amplitude for the two bosons, which of course is closely related to the indistinguishability of the two processes leading to this final state, as shown in Fig. 1. The probability of such an event is then given by the square of A_{12}, integrated over all source points, which may be written as a function of the four-momentum difference $q \equiv p_2 - p_1$,

$$\mathcal{P}_{12} = \int d^4x\, d^4y\, |A_{12}|^2 \rho(x)\rho(y) = 1 + |\tilde{\rho}(q)|^2 \equiv C_2(q)\,, \tag{2}$$

assuming that the production amplitudes are independent of momentum and *that they are mutually incoherent.* The case where the amplitudes depend both on momentum and position will be treated in Section 3, while the precise meaning of "mutually incoherent" will be developed in Section 4. For now, we will simply state that the second condition is roughly equivalent to assuming that the production amplitude for the "second" pion is independent of that for the "first". Those who desire a more detailed explanation may find one in Section 4.

Equation 2 is the fundamental HBT result. It states that the two-particle counting rate is proportional to the square of the Fourier transform of the source distribution. The Fourier transform is performed with respect to the relative (four)-momentum of the particles, with each component of relative momentum containing information of the source distribution in the conjugate spatial variable. The range of the enhancement for the i-th component q_i is of order $\hbar/R_i$, as expected from uncertainty principle arguments. The normalization of $\rho(r)$ enforces the condition $C_2(q=0) = 2$, an appropriate result: The probability of finding two like-pions in the same state is 2! times greater than for unlike particles. (Although by no means apparent to the uninitiated, this result is directly related to our assumption of source chaoticity.)

The function $C_2(q)$ is the correlation function for pair detection. In terms of ideal observables, it is given by

$$C_2^{ideal}(q) = \frac{\langle n \rangle^2}{\langle n(n-1) \rangle} \cdot \frac{\dfrac{dn}{d\vec{p}_1 d\vec{p}_2}}{\dfrac{dn}{d\vec{p}_1}\dfrac{dn}{d\vec{p}_2}}\,, \tag{3}$$

(The factors of $\langle n \rangle$, etc. follow from the normalization of one and two-particle inclusive distributions.) In practice, the correlation function is never constructed from a ratio given by Eq. 3, since it requires knowing the full six-dimensional two-particle distribution. Instead a projection[7] $A(\vec{p}_1, \vec{p}_2) \to A(q)$ of the actual data is made, then compared to a similar projection of some background sample $B(q)$, so that the empirical correlation function is given by

$$C_2^{empirical}(q) = \frac{A(q)}{B(q)} \quad . \tag{4}$$

Typical prescriptions for generating $B(q)$ involve using unlike sign pions, or creating artificial pairs by combining single tracks from different events. The extent to which such procedures allow the identification $C_2^{empirical} \stackrel{?}{=} C_2^{ideal}$ is an open issue that must be addressed experiment-by-experiment. Possible sources of systematic error result from acceptance-induced residual correlations[8], implicit event-class averaging[9], and exclusion of single-pion events from $B(q)$[10]. Space does not permit discussion of these effects; the reader is referred to the original references. However, it is worth noting that the difference between an analytic C_2^{ideal} and an experimentally constructed $C_2^{empirical}$ becomes moot *if one can apply the same experimental treatment to a theoretically-generated set of events.* For example, such a set of events might result from a cascade code, which are then "correlated" according to the methods presented in Section 3.

2.2 Trivial Aspects of Correlation Function Parameterization

Let us take Eq. 2 at face value and assume that the correlation function measures the *four*-dimensional Fourier transform of the source *spatial* distribution. (We will see in Sections 2.3 and 2.4 that the italicized words have important caveats associated with them.) The typical method of analyzing HBT data then proceeds as

- Pick some favorite form for $\rho(r;R)$, with a set of source parameters generically denoted by "R".
- Fourier transform it: $\rho(r;R) \to \tilde{\rho}(q;R)$.
- Vary R to find the best fit of $C_2^{empirical}$ to the form $1+\lambda|\tilde{\rho}(q;R)|^2$ (What little I have to say about λ may be found in Section 4.)
- Identify the resulting value(s) for R as the source parameter(s).

This should be straightforward enough, but runs the risk that the careful experimenter will have his or her value for R quoted by those who have not carefully examined the preceeding steps, and thus proceed in ignorance of the relationship between R and some standard measure of a source radius, e.g., $\sqrt{\langle r^2\rangle}$.

To understand how such an unfortunate state of affairs could (and has) come about, consider the difference between parameterizing Gaussians in one and three dimensions. In one dimension, we have

$$\rho(x;\sigma) \sim e^{-x^2/2\sigma^2} \quad \Leftrightarrow \quad \tilde{\rho}(q;\sigma) \sim e^{-(q_x\sigma)^2/2} \tag{5}$$

$$\Rightarrow \quad |\tilde{\rho}(q;\sigma)|^2 = e^{-(q_x\sigma)^2} \quad , \tag{6}$$

with $\langle x^2\rangle = \sigma^2$. If we extend this to three dimensions, we have

$$\rho(\vec{r};\sigma) \sim e^{-|\vec{r}|^2/2\sigma^2} \quad \Leftrightarrow \quad \tilde{\rho}(\vec{q};\sigma) \sim e^{-|\vec{q}|^2\sigma^2/2} \tag{7}$$

$$\Rightarrow \quad |\tilde{\rho}(\vec{q};\sigma)|^2 = e^{-|\vec{q}|^2\sigma^2} \tag{8}$$

but now with $\langle \vec{r}\cdot\vec{r}\rangle = 3\sigma^2$. The fact that in three dimensions the rms radius and the fitted value of σ differ by $\sqrt{3} \approx 1.7$ apparently motivated some authors to sneakily eliminate the factor of 2 in Eq. 8: $\rho(\vec{r};\sigma) \to e^{-r^2/R^2}$, so that $\langle \vec{r}\cdot\vec{r}\rangle = \frac{3}{2}\sigma^2$, thereby "improving" the correspondence between σ and the (3-d) rms. Such a rescaling would be acceptable if it were restricted to the three-dimensional case. However, it creates problems when the corresponding functional form for C_2 is extended to asymmetric sources, i.e.,

$$|\tilde{\rho}(\vec{q};\sigma)|^2 = e^{-\frac{1}{2}|\vec{q}|^2R^2} \to e^{-\frac{1}{2}[(q_xR_x)^2+(q_yR_y)^2+(q_zR_z)^2]} \quad . \tag{9}$$

Now the spurious 2 in the *squared* Fourier transform produces *one*-dimensional fit parameters that are $\sqrt{2}$ larger than their associated rms values: $\sqrt{\langle x^2\rangle} = \sqrt{2}R_x$, etc.

While, as indicated by this sub-section title, this ought to be a trivial problem, it has resulted in a non-trivial amount of confusion concerning the geometric interpretation of extracted HBT radii. Suffice it to say that some fraction (perhaps of order $(\sqrt{2}-1) \sim 40\%$) of the evidence for anomalously large radii that have been reported at various times results simply from this. Since there are truly subtle effects that must be addressed in HBT measurements, it is imperative that this non-subtle one be eliminated. To summarize:

- When fitting to Gaussians, whether in one, two or three dimensions, the general form $|\tilde{\rho}(q;R)| = e^{-(qR)^2}$ should be used (i.e., without a $\frac{1}{2}$ in the exponent).
- Experiments differ in the acceptance for different components of $\vec{q}$. The acceptance limits and their effects on the extracted parameters should be clearly presented.
- Since the Gaussian is at best an approximation to the true spatial distribution, experimenters and theorists alike should carefully qualify statements concerning the comparison to "geometric" values.

A recent example that attempts to adhere to these criteria within a stringent length limit may be found in Ref. [11].

2.3 Dimensionality of the Fourier Transform

One feature of Eq. 2 that survives in the more rigorous derivation found in Section 3 is the dependence of the correlation function on a four-dimensional integration of the source density distribution (squared) against $e^{iq(x-y)}\,d^4x\,d^4y$. Since one is integrating over the full space-time distribution, it is tempting to conclude that the resulting $|\tilde{\rho}(q)|^2$ is a truly four-dimensional object. Like many tempting things, it is wrong.

To understand this, first consider the region of integration we would use to invert $\tilde{\rho}(q)$ assuming q refers to the four-momentum of a single particle. Then, since the particle is by definition on its mass shell, any inverse transform of $\tilde{\rho}(q)$ must enforce this condition by requiring $d^4q \rightarrow \delta(q^2 - m^2)d^4q$. This implies that, rather than freely roaming the $dq_0\,d^3q$ "plane" illustrated in Figure 2b, we are restricted to the mass-shell "line" $q_0 = \sqrt{|\vec{q}|^2 + m^2}$ of Figure 2a. Intermediate between these extremes is the case where q is the momentum difference between two on-shell particles, as illustrated in Figure 2c. If we define $q_0 \equiv |E_2 - E_1|$, $\vec{q} = \vec{p}_2 - \vec{p}_1$, then simple four-vector algebra shows that we must have $q_0 \leq |\vec{q}|$, corresponding to the shaded region of Figure 2c. This restriction has an important consequence. Another simple calculation shows that

$$q_0 = \vec{q}\cdot\vec{\beta}^* \quad , \tag{10}$$

where the two-particle center-of-mass velocity $\vec{\beta}^*$given by

$$\vec{\beta}^* \equiv \frac{\vec{p}_1 + \vec{p}_2}{|E_1 + E_2|} \quad . \tag{11}$$

Physically, this relation means that the source parameter conjugate to q_0 (the temporal distribution of emission points) is always measured in conjunction with the *spatial* distribution of emission points *along* the direction of motion of the pair. This statement is equivalent to the observation that the restricted region in Figure 2c.) prevents one from exploring the full q_0 structure of the correlation function at a fixed (small) value of $|\vec{q}|$. To demonstrate this explicitly, consider a source distribution given by

$$\begin{aligned} \rho(\vec{r},t) &\sim e^{-|\vec{r}|^2/2R^2 - q_0^2/2\tau^2} & (12)\\ \Rightarrow \tilde{\rho}(q) &\sim e^{-(|\vec{q}|R)^2/2 - (q_0\tau)^2/2} & (13)\\ &\sim e^{-(|\vec{q}|R)^2/2 - (\vec{q}\cdot\vec{\beta}^*\tau)^2/2} \quad , & (14) \end{aligned}$$

which shows that for $\vec{q} \perp \vec{\beta}^*$, one measures only R^2, while for $\vec{q} \parallel \vec{\beta}^*$, one sees $R^2 + (\vec{\beta}^*\cdot\tau)^2$.

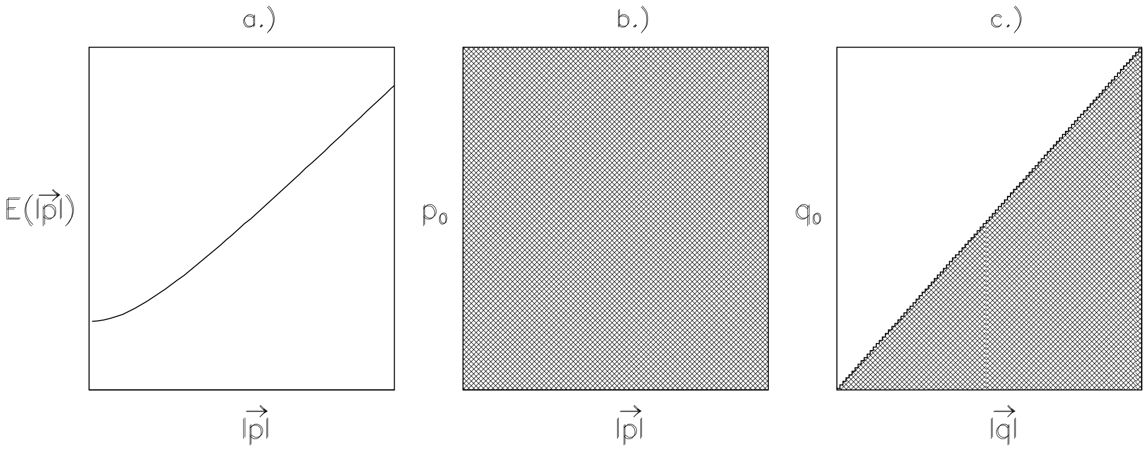

Figure 2: Allowed regions of integration for a.) a particle on its mass shell, b.) an arbitrary four-vector p, and c.) relative momentum difference between two (on shell) particles.

2.4 Correlation Functions Measure *Relative* Separation

One usually states that interferometry measures the squared-Fourier-transform of the original source distribution. This is certainly supported by the argument leading to Eq. 2, but any Fourier-transformer knows that a squared F.T. is really a measure of *relative* separation in the source:

$$|\tilde{\rho}(q)|^2 = \int e^{iqr}\, dr \Big\{ \int \rho(R + \tfrac{1}{2}r)\rho(R - \tfrac{1}{2}r)\, dr \Big\} \tag{15}$$

(assuming $\rho(x)$ is real). Furthermore, in the case of particle interferometry, the relative separation is the quantity of physical significance, while the passage to the squared form is a mathematical convenience. (This is made particularly clear by Koonin [12], who shows that the correlation depends on the overlap between the two wave-packets, one centered at $(\vec{r}_2, t_2)$, the other at $(\vec{r}_1 + \vec{\beta}^*(t_2 - t_1), t_2)$, plus the same function with $1 \leftrightarrow 2$. This same result has been obtained in a Wigner formalism by Gong, Bauer and Pratt[13].)

It is an unhappy fact that relative separation is much less shape-sensitive than absolute separation. While this was described as a "well-known property of the Fourier transform" in Goldhaber, Goldhaber, Lee and Pais's (GGLP) first application of interferometry to particle physics[14], this property is often-overlooked, leading to false expectations for the sensitivity of HBT measurements to subtle features of the source distribution[15]. For example, GGLP note that the correlation function for a uniform spherical density of radius R differs from our Eq. 8 by 2% or less everywhere provided $R = 2.15\sigma$.

Let us try to understand this result by comparing the distributions in coordinate space. For a Gaussian, the calculation of the distribution for $r \equiv |\vec{r}_1 - \vec{r}_2|$ is trivial, giving $f_{Gaus}(r)\, d^3r \sim e^{-r^2/4\sigma^2}\, d^3r$. The corresponding calculation for the relative separation in a uniform spherical distribution of radius R, which relies upon a "well-known" trick[16], gives

$$f_{unif}(r)\, d^3r \sim \begin{cases} [1 - \frac{3}{2}(\frac{r}{2R}) + \frac{1}{2}(\frac{r}{2R})^2]\, d^3r & \text{for } r < 2R \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

Figure 3 shows that the differences in spatial separation between the Gaussian and uniform distribution are also of order a few percent, again provided $R = 2.15\sigma$ (note that the apparent factor of two difference between R and σ is an artifact of the parameterization, the ratio of the rms values is given by

$$\frac{(\mathrm{rms})_{unif}}{(\mathrm{rms})_{Gaus}} = \frac{\sqrt{\frac{3}{5}}R}{\sqrt{3}\sigma} = 0.96 \quad . \tag{17}$$

The fact that a uniform distribution tends to "look" Gaussian is not restricted to this case. It is straightforward to show that, for sources with an inversion symmetry $\rho(\vec{r}) = \rho(-\vec{r})$, there is a close connection between the squared-Fourier-transform and the central limit theorem. As always, the relevant question then becomes "How quickly does my sampling of the parent distribution relax towards a Gaussian?"

2.5 Lorentz Properties of Correlation Functions

As made clear in Section 2, the experimenter is always faced with a choice when projecting the distribution of actual pairs into some sort of relative momentum variables, generally described as $A(\vec{p}_1, \vec{p}_2) \rightarrow A(q, P)$, where q is some generic relative momentum variable and P is the corresponding total momentum variable for the pair. Naively, one might think that $(\vec{p}_1, \vec{p}_2) \rightarrow (\vec{q} \equiv \vec{p}_1 - \vec{p}_2, \vec{P} = \frac{1}{2}(\vec{p}_1 + \vec{p}_2))$ is the logical choice. However, this method has at least two difficulties:

- The limited number of events obtained in a typical HBT experiment is not adequate to map out the full six-dimensional space implied by this procedure.
- It is frame-dependent, while the correlation function as expressed in Eq. 3 is Lorentz invariant.

Some confusion has resulted from the fact that there is a relative momentum variable that has the tempting appeal of simultaneously solving both problems (recall cautionary words concerning temptation in Section 2.3). That variable, of course, is the invariant relative momentum of the pair, ${Q_{inv}}^2 \equiv (\vec{p}_1 - \vec{p}_2)^2 - (E_1 - E_2)^2$. Thus, starting from the observation that the *ideal* correlation function is a Lorentz invariant, one argues that

$$C_2^{empirical}(\vec{p}_1, \vec{p}_2) \equiv \frac{A(\vec{p}_1, \vec{p}_2)}{B(\vec{p}_1, \vec{p}_2)} \rightarrow \frac{A(Q_{inv})}{B(Q_{inv})} \equiv C_2^{empirical}(Q_{inv}; R_Q) \quad , \tag{18}$$

where I have denoted in the last step R_Q as the spatial parameter conjugate to Q_{inv}. There can be no doubt that this method is tremendously useful to the experimenter with limited statistics. Where doubt can (and should) arise is in the interpretation of R_Q. Briefly put, R_Q does measure an invariant length– but it's not the source dimension.

Several arguments will be presented demonstrating the nature of the effect. Before proceeding, it is crucial to note that these arguments all obtain in the simplest possible case, i.e., where we are attempting to determine the rest length(s) of a static source. At the end of this section, I will present conclusions concerning Q_{inv} and its applicability to dynamic source models.

2.5.1 Pure Empiricism

It is a simple matter to create pairs of particles correlated according to some simple source distribution, then analyze the resulting "data" in terms of Q_{inv} . In the course of analyzing E802[11] and E859[17] data for Si+A→ $2\pi^{\pm} + X$, we were led to consider the form

$$C_2(|\vec{q}|, q_0) = 1 + e^{-(|\vec{q}|R)^2 - (q_0\tau)^2} \quad , \tag{19}$$

with $R \sim \tau = 2$ fm. However, initial measurements of the (quite rare) 2K$^+$pairs did not permit such a (two-dimensional) fit, so we decided to perform a three-step procedure:

1.) Analyze both the kaon and the pion data using the one-dimensional Q_{inv} form $C_2(Q_{inv}; R_Q) = 1 + e^{-(Q_{inv}R_Q)^2}$.

2.) "Calibrate" the Q_{inv} measurement from the pion data, where the much higher statistics permitted extraction of both of both R_Q and (R, τ).

3.) Use the information obtained in Step 2 to infer the likely values of R and/or τ for the kaons.

However, Monte Carlo simulations of this effect in our spectrometer produced the following result: While both pion and kaon pairs correlated according to Equation 19 are well fit to the $C_2(Q_{inv}; R_Q)$ form, the values of R_Q obtained for pions are *double* those found for kaons: $R_Q(2\pi) \approx 2R$, while $R_Q(\mathrm{2K}) \approx R$. In the next section we will explain the origin of this difference. Here, we merely wish to note that 1.) Clearly R_Q depends on something other than source dimensions (R, τ) (otherwise it would be the same for pions and kaons) and 2.) Correctly applying the empirically determined correction factors greatly reduces the observed experimentally observed difference between the pion and kaon radii[17].

2.5.2 Elementary Explanation I

Recall that interferometry follows from the uncertainty relationship, which in any frame implies $qr \sim \hbar$, where "q" now denotes the region of observed enhancement in C_2. Then, the transformation property of HBT radii is contained in the statement

$$|\vec{q}|R \sim \hbar \sim Q_{inv}R_Q \quad , \tag{20}$$

provided that there is a frame where Q_{inv} behaves as a "real" momentum, i.e., one derived from a canonical commutation relation. Fortunately, there is such a system, for Q_{inv} represents the relative momentum between two particles *in their mutual center-of-mass frame.* Contained in this observation is most of the subtlety that arises in the Q_{inv} formalism:

- R_Q measures the source "radius" as observed in the *from* the two-particle rest frame (more properly, it's not the radius, but the distribution of relative separation in the source).

- The transformation property of R_Q back to the source rest frame depends on the orientation of $\vec{q}$ relative to $\vec{\beta}^*$.

- R_Q necessarily contains an ensemble average over all the distribution of pair velocities used to measure the correlation function. As a result, the relationship between R_Q and a radius in the source rest frame is also dependent on the distribution of accepted pairs for a given measurement.

To appreciate the range of possible effects, consider two cases:

- $\vec{q} \perp \vec{\beta}^*$:
  Then $Q_{inv} = \vec{q}$, and $R_Q = R$.

- $\vec{q} \parallel \vec{\beta}^*$:
  Then $Q_{inv} = |\vec{q}|\sqrt{1 - |\vec{\beta}^*|^2} \equiv |\vec{q}|/\gamma_{pair}$, so that $R_Q = \gamma_{pair} R$.

It is this last instance that is counter-intuitive: In the pair rest-frame, the receeding source appears Lorentz-extended, *not* Lorentz-contracted, in the direction of the pair relative velocity. The "2" that appears in the empirical result for pions mentioned in Section 2.5.1 $R_Q(2\pi) \approx 2R$ is then simply the average value of the Lorentz γ factor of the E859 pion pairs with respect to the (assumed stationary) source. The corresponding factor for kaons is much closer to unity, since the accepted kaons have comparable momenta to the pions, and hence lower velocities.

2.5.3 Elementary Explanation II

In the previous section, an elementary quantum mechanical argument produced a result (Lorentz-extension) in contradiction with elementary relativity. Or is it? Consider the origin of Lorentz-contraction in special relativity: Objects appear contracted in the direction of motion, *when their lengths L are measured by multiplying their velocity β by the time Δt required to pass a stationary point in the observer's frame.* Since the two events in the object's rest frame are separated by an interval $\Delta t' = L_o/\beta, \Delta x' = -L_o)$, a Lorentz transformation then gives $\Delta t = L_o/\beta\gamma \Rightarrow L = L_o/\gamma$.

This is not how interferometry measures lengths! Instead, recall that the correlation is "induced" by the relative separation at the time the "second" particle is emitted. In this case, the interval being measured[18] is of the form $(\Delta t' = 0, \Delta x')$. Performing a Lorentz transformation on $\Delta x'$ then gives $\Delta x = \gamma \Delta x'$. Physically, the measurement process corresponds to determining the length of an object by measuring the time interval between the arrival of two simultaneously emitted particles, one from the front, the other from the rear of the object. Although (particle) interferometry processes this information in the momentum, not time, domain, the end result is the same: A Lorentz extension of lengths parallel to the pair velocity.

2.5.4 A Truly Invariant Formulation for C_2

It should be clear by now that R_Q has an invariant value: It measures the source distribution as seen from the ensemble of *pair* rest-frames. Since the Lorentz effects occur relative to the assumed rest lengths of the source, it is convenient to have a formalism in which the source velocity appears as an explicit parameter. Such an expression was derived by Yano and Koonin[19] some time ago. They noted that a correlation function resulting from a source density of the of the form given in Eq. 19, while not Lorentz invariant, could be made so by explicitly introducing the source four-velocity u_s:

$$C_2(|\vec{q}|, q_0) = 1 + e^{-(|\vec{q}|R)^2-(q_0\tau)^2} \rightarrow 1 + e^{-(QR)^2-[R^2+\tau^2](q\cdot u_s)^2} \quad . \tag{21}$$

This form is obviously invariant, and reduces to the non-invariant one when $u_s \equiv \gamma_s(1, \vec{\beta}_s) = (1, \vec{0})$, i.e., when in the source rest-frame. Thus, even for a static source, the correlation function depends on the relative orientations of $\vec{q}$, $\vec{\beta}^*$and $\vec{\beta}_s$. For the simple case where $\vec{\beta}_s = 0$ the correlation function depends on Q_{inv} , $|\vec{\beta}^*|$ and θ^*, the angle between $\vec{\beta}^*$ and $\vec{q}$:

$$(|\vec{q}|R)^2 + (q_0\tau)^2 = Q_{inv}^2 \cdot \frac{R^2 + (|\vec{\beta}^*|\tau)^2\cos^2\theta^*}{1 - |\vec{\beta}^*|^2\cos^2\theta^*} \quad , \tag{22}$$

which contains implicitly all the Lorentz effects discussed here. The more general case for non-zero $\vec{\beta}_s$ has been discussed by Makhlin and Sinyukov[20, 21].

2.5.5 When Is Q_{inv} the Right Variable?

The preceeding arguments should induce a certain wariness (weariness?) in those desirous of using a Q_{inv} parameterization. I will conclude this section with examples of where it may in fact be the appropriate relative momentum variable, but would first like to indicate a case where it is even more misleading than noted above. Suppose one repeats the $(R, \tau) \rightarrow R_Q$ calibration exercise described in Section 2.5.1 for varying values of R and τ. For $R \lesssim \tau$ one finds that the resulting Q_{inv} correlation function is roughly Gaussian, but for $R > \tau$, the input Gaussian in $|\vec{q}|, q_0$ is sufficiently distorted that the correlation function appears *exponential.* An example of this is shown in Figure 4 for the E859 acceptance and $R = 3$ fm, $\tau = 6$ fm. The resulting distribution in Q_{inv} favors an exponential fit to

$$C_2(Q_{inv}) = 1 + \lambda e^{-2Q_{inv}R_E} \tag{23}$$

with $R_E = 4.44 \pm 0.02$ and $\lambda = 1.41 \pm 0.01$ (the factor of 2 in the exponent is an empirical factor necessary for $R_E \sim R_Q$, and as such gives some idea of the magnitude of the systematic distortion present in $C_2(Q_{inv})$). This result is not peculiar to one experiment's acceptance; similar results have been obtained in simulations of Bose-Einstein measurements at RHIC.

Roughly speaking, the transition to an "exponential" from a Gaussian when $R > \tau$ may be understood as follows: As apparent from Eq. 22, the component of $\vec{q}$ parallel to $\vec{\beta}^*$is increased by $|\vec{\beta}^*|\tau$, which is in turn extended by a Lorentz factor. The two components of $\vec{q}$ perpendicular to $\vec{\beta}^*$are unaffected. The correlation function therefore develops a "two"-component structure, with the "parallel" radius 2-4 times larger than the "perpendicular" radius. For typical acceptances, it is an unhappy fact that the averaging over $\vec{q}$ angles with respect to $\vec{\beta}^*$contained in the Q_{inv} form causes this "two"-component form to appear approximately exponential.

Having said that, it is crucial to note that there is an entire class of models where *a (sole) dependence on Q_{inv} emerges naturally.* This occurs whenever the source distribution is a function of the proper time (only): $\rho(\vec{r}, t) \rightarrow \rho(t^2 - |\vec{r}|^2)$. Bowler[22] has provided an explicit calculation in a one-dimensional string model that both demonstrates this and explores the subtle interplay between geometry and dynamics. Here we content ourselves with a sketch only of the kinematic transformation used by Bowler:

Suppose that source is one-dimensional, and depends only on the proper time, so that the Fourier transform is

$$\tilde{\rho}(q) = \int e^{iqr} \rho([t^2 - z^2]/\tau_o^2)\, d^2 r \quad , \tag{24}$$

where τ_o is some parameter that sets the scale in proper time. Introducing the light-cone variables

$$u_\pm \equiv \frac{t \pm z}{\sqrt{2}} \tag{25}$$

$$q_\pm \equiv \frac{q_o \mp q_z}{\sqrt{2}} \tag{26}$$

one has

$$\tilde{\rho}(q) = \int e^{i(q_+ u_- + q_- u_+)} \rho(2u_+ u_- / \tau_o^2)\, du_+ du_- \tag{27}$$

$$= \tilde{\rho}(\,1/(q_+ q_- \tau_o^2)\,) \quad , \tag{28}$$

where the last line follows from dimensional considerations alone. Since $q_+ q_- = \frac{1}{2}(q_0^2 - q_z^2) \equiv Q_L^2$, it is clear that the correlation function depends on Q_L^2. This is as it should be– a source function which depends only on proper time describes physics which depends on the time as seen in the rest frame of each individual particle (for example, longitudinal growth). It is then plausible that the correlation function depends only on the relative momentum seen in the rest frame of the various pairs, which is Q_{inv} .

While the extension of the above argument to three spatial dimensions is straightforward, the physics involved is not. Clearly, the presence of a jet axis[22, 23], or a deep-inelastic scatter[24], or an e^+e^- annihilation[25] guarantees the presence of a quasi-one-dimensional axis along which (proper-time dependent) hadronization proceeds, thereby making Q_L^2 an appropriate variable. In central A+A collisions, the peripheries of the interaction regions are causally disconnected from each other. It remains an open question whether the subsequent re-interactions of the produced particles are sufficient to lead to a hadronization region which depends only on the quantity $t^2 - |\vec{r}|^2$. Interferometry is the logical candidate to answer that question.

3 The Wigner Function Formulation for Correlation Physics

3.1 Introduction

The discussion of the previous section clearly demonstrates that the nature and parameterization of the particle sources in heavy ion collisions is by no means obvious from first principles. Suppose, as an alternative, that one wishes to model the complicated dynamics of a heavy ion collision via a Monte Carlo cascade code. Such a model, which by its very nature must deal with probabilities, not amplitudes. A natural question to ask is how might we take the resulting description of the collision and infer the expected Bose-Einstein correlations? (For these purposes, we regard the cascade code as an essentially classical calculation of the single-body phase space distribution functions $f(\vec{r}, \vec{p})$, along with all *dynamical* correlations between particles.)

The answer, as first noted by Pratt[26], is formulated in terms of the Wigner function, which is actually a mixed representation of the density matrix for the system. If that sounds unfamiliar, the reader is referred to the appendix, which should supply a reasonable introduction to this formulation of quantum mechanics. Here, we show how the one and two-particle momentum distribution functions may be written in this language. No claim is made towards originality; this section is essentially a translation of the rather compact notation of Ref. [27] into a more pedagogical development, based in part on the material of the appendix.

3.2 Single-Particle Distribution

We begin with a generally valid reformulation of quantum mechanics that has an (apparently) classical form: The single-particle momentum density evaluated at some final momentum p is given by

$$P_1(p) = \sum_i \mathcal{P}(i) \, | \int d^4x \langle p|U(t \to \infty|x\rangle\langle x|i \rangle|^2 \quad . \tag{29}$$

Here $U(t)$ is the time-evolution operator, x represents any intermediate space-time point, $|i\rangle$ is the initial state of the system[28], and $\mathcal{P}(i)$ is the probability of being in that state. By explicitly writing out the complex square, this becomes

$$P_1(p) = \int d^4x \, d^4y \rho(x,y) \Pi^*(p;x,y) \quad , \tag{30}$$

where the one-body density matrix is given by

$$\rho(x,y) \equiv \sum_i \mathcal{P}(i)\langle x|i\rangle\langle i|y\rangle \equiv \langle x|\rho|y\rangle \quad , \tag{31}$$

and

$$\Pi^*(p;x;y) \equiv \langle y|U^\dagger|p\rangle\langle p|U|x\rangle \quad . \tag{32}$$

Thanks to an obscure property of the Wigner transform, Eq. 30 can be re-written in terms of the corresponding Wigner functions as

$$P_1(p) = \int d^4r \, \frac{d^4k}{(2\pi)^4} \, \rho_W(r,k) \Pi_W^*(p;r,k) \quad , \tag{33}$$

which resembles a classical integral of a one-body (phase-space) source density $\rho_W(r,k)$ against the probability Π_W^* for particles emitted at r with momentum p to propagate and be detected at infinity with momentum p.

For the reader unhappy with the magical appearance of Wigner functions in Eq. 33, it is worth noting that they appear naturally in the context of Eq. 30 once we make the identification

$$\Pi^*(p;x,y) = e^{ip(x-y)} \Rightarrow P_1(p) = \int d^4x \, d^4y \rho(x,y) e^{ip(x-y)} = \int \rho_W(r,p) \, d^4r \quad . \tag{34}$$

For the single-particle case, Π_W is trivially given by

$$\Pi_W(p;r,k) = (2\pi)^4 \delta(p-k) \quad , \tag{35}$$

which of course implies that

$$P_1(p) = \int d^4r \, \rho_W(r,p) \quad . \tag{36}$$

3.3 Two-Particle Distribution

We can now proceed formally with the two-body case:

$$
\begin{aligned}
P_2(p_1,p_2) &= \sum_i \mathcal{P}(i)\,|\int d^4x\,d^4y\,\langle p_1,p_2|U(t\to\infty|xy\rangle\langle xy|i\,\rangle|^2 & (37)\\
&= \sum_i \mathcal{P}(i)\int d^4x\,d^4y\,d^4x'\,d^4y'\,\langle i|x'y'\rangle\langle x'y'|U^\dagger|p_1,p_2\rangle\langle p_1,p_2|U|xy\rangle\langle xy|i\rangle \\
&= \int d^4x\,d^4y\,d^4x'\,d^4y'\,\rho(xy;x'y')\Pi^*(p_1p_2|xy;x'y') \quad . & (38)
\end{aligned}
$$

Note that in the above equations ρ is now the *two*-body density matrix, as evidenced by its argument count. At this point, we could either a.) Fervently hope that the property of Eq. 33 extends to the two-body case or b.) Think about the physics. Following the authors of Ref. [27], we will opt for b.). That is, we require that the particles are emitted independently (as appropriate for a chaotic source), so that

$$\rho(xy;x'y') \approx \rho(x,x')\cdot\rho(y,y') \quad . \qquad (39)$$

(I will comment further on this assumption at the end of this section.) We also note that the symmetrization requirement appears in the propagator, i.e.,

$$
\begin{aligned}
\Pi^*(p_1p_2|xy;x'y') &= \langle x'y'|U^\dagger|p_1,p_2\rangle\langle p_1,p_2|U|xy\rangle \\
&= [\frac{1}{\sqrt{2}}(e^{ip_1y_1+ip_2y_2}+e^{ip_2y_1+ip_1y_1})]\times \\
&\quad [\frac{1}{\sqrt{2}}(e^{-ip_1x-ip_2y}+e^{-ip_2x-ip_1x})] \quad . & (40)
\end{aligned}
$$

If we substitute Eqs. 39 and 40 into 38, and make transformations to the relevant relative coordinates when doing the integrations, it is straightforward to show that

$$P_2(p_1,p_2) = \int d^4x\,d^4y\,[\rho_W(x,p_1)\rho_W(y,p_2)+\rho_W(x,K/2)\rho_W(y,K/2)\cos q(x-y)] \quad , \qquad (41)$$

where the four-vectors q and K are defined by $q\equiv p_2-p_1$ and $K=\frac{1}{2}(\,E_1+E_2,\ \vec{p}_1+\vec{p}_2\,)$. The form of Eq. 41 is just what we would have obtained with the usual naive finger-physics HBT derivation, *except* that the Wigner functions are evaluated off mass-shell in the "cross" term(!). We can get some feeling for this surprising result by noting that this result follows from our desire to formulate the problem in terms of a classical phase space distribution function $\rho(r,p)$. Such a treatment is treats position and momentum symmetrically. The quantum mechanical path ambiguity implicit in the (spatial) symmetrization of the two-body wavefunction now treats momentum and space symmetrically: There is confusion as to the spatial origin of each boson, while at the same time we are unable to say which source contributed to the total momentum and energy of the final detected pair.

It is now straightforward to obtain the correlation function:

$$C_2(p_1,p_2)\equiv\frac{P_2(p_1,p_2)}{P_1(p_1)\cdot P_1(p_2)} = 1+\frac{|\int e^{iqx}\rho_W(x,K/2)\,d^4x|^2}{[\int\rho_W(x,p_1)\,d^4x]\cdot[\int\rho_W(x,p_2)\,d^4x]} \quad . \qquad (42)$$

So long as we interpret ρ_W as the Wigner density, this result remains a strictly correct quantum mechanical result (modulo the plane wave and factorization requirements we have made along the way). The useful *ansatz* is that, with the identification $\rho_W(r,p)\to f(r,p)\equiv$ a classical distribution function, it is our best guess for imposing

Bose-Einstein correlations on a source whose emission probabilities depend on momentum and position.

As a specific example, we consider a source given by

$$\rho(\vec{r},\vec{p}) \sim \exp\{-\frac{1}{2(1-\alpha^2)}[|\vec{r}|^2/R^2 - 2\alpha\vec{p}\cdot\vec{r}/(RP) - |\vec{p}|^2/P^2]\} \quad . \tag{43}$$

Both $\vec{r}$ and $\vec{p}$ are simple Gaussians for this distribution, with rms values given by R and P respectively. The parameter α ($|\alpha| < 1$ controls the extent to which $\vec{r}$ is correlated with $\vec{p}$. The reader is encouraged to verify these relations, and to show that if one takes Eq. 43 as the Wigner density in Eq. 42, it leads to a source radius R_W given by

$$R_W^2 = R^2[(1-\alpha^2) - \frac{1}{(2PR)^2}] \quad . \tag{44}$$

Even in the case where $\alpha = 0$, the Wigner formalism leads to a reduction of the measured radius. (Note that it gives nonsense if one attempts to localize the source in phase space more than allowed by the uncertainty relationship.) For non-zero α, a further reduction occurs. Its physical origin is very simple: when $\vec{r}$ and $\vec{p}$ are correlated, pairs close in momentum space come from regions that are also close in coordinate space. (This simple form of course neglects the complications of the energy variation and the mass shell conditions, and as such does not show the usual dependence of the R_W on $\vec{K}$. Also, the fact that there are values for α that lead to $R_W^2 < 0$ is troubling! It is not obvious to me if the uncertainty relations protect us from this non-physical regime.)

Even the not-so-careful reader may have been struck by the drastic factorization assumption used in Eq. 39 to write the two-body density as the product of two single-particle densities. Unfortunately, this does represent state-of-the-art in cascade code simulations. That is to say, the correlations due to Bose-Einstein symmetrization are put in "after" all the dynamics has run its course. The truly omniscient reader will have noted that this assumption is the density-matrix embodiment of the (yet-to-come) chaoticity requirement given in Eq. 56. The extent to which the bosonic nature of mesons plays a role in the actual *dynamics* of dense hadronic matter is at present unknown, although some have speculated that it is responsible for the low-p_T enhancement of pion spectra [29, 30], and that the relevant factor in the distortion of the two-body correlation function by higher-order correlations is the number of bosons per unit phase space cell[31]. Perhaps some insight in these matters may be gained from ongoing efforts to incorporate Bose factors into transport equations[32].

3.4 The Promise of Monte Carlo Models

Suppose one has struggled through the Wigner formulation, and used it to create correlated pairs of pions from a cascade code[33]. What might one hope to see? Only recently have such methods been applied to heavy ion collisions at AGS and SPS energies[34]. Here I present a small glimpse of the information content potentially available[35] for the specific case of the ARC cascade code for collisions at AGS energies[36]. Plotted in the top panel of Figure 5 is the x-distribution of the final strong interactions for all pions created in a central Si+Au collision. (Here x is an arbitrary direction perpendicular to the collision axis.) One sees a well-defined Gaussian distribution combined with a long tail presumably due to long-lived resonance decays (the magnitude of this tail is in rough agreement with the estimates provided by the authors of Ref. [11]). While the distribution is symmetric about the z-axis, it is no longer so if one observes it through a typical experimental aperture. This is shown in the middle panel of Figure 5, where the effect of restricting the angular range $|\phi| \equiv |\tan^{-1}(y/x)| < 10°$ is seen to both decrease the width and to shift the mean x value away from zero. This effect results from the definition of the plotted quantity: It is very unlikely for those pions "born" on the $-x$ side of the distribution to make their way to a detector sited on the $+x$ side without reinteracting. But by definition they are not allowed to reinteract, since the figure plots what interferometry measures: the distribution of last strong interactions. The bottom panel of Figure 5 shows that this effect becomes even more noticeable when a realistic estimate of the E859 acceptance is used.

It is clear that ARC predicts that any limited-aperture experiment will "see" only one side of the nucleus. In fact, since this results from a correlation between the directions of $\vec{p}$ and $\vec{r}$ for final-state pions, the same would be true in a 4π experiment–(but with better statistics). It is amusing to note that this produces one of the few truly useful applications for *three* pion interferometry: Suppose we take the extreme case where the "true" distribution is a uniform sphere, and the "biased" distribution is a uniform *half*-sphere. Can we tell them apart via interferometry? Naively, one would say "Of course", but the reader who has absorbed the lesson concerning relative separations in Section 2.4 may be (properly) skeptical. The Figure 6a shows that two-particle interferometry gives virtually identical correlation functions for a sphere

$$C_2(q) = 1 + |\frac{3}{x^3}[\sin x - x\cos x]|^2 \text{ with } x \equiv qR_S \quad , \tag{45}$$

as for a hemisphere

$$\begin{aligned} C_2(q) &= 1 + |\tilde{\rho_R} + i\tilde{\rho_I}|^2 \text{ with } x \equiv qR_H \text{ and} & (46)\\ \tilde{\rho_R} &\equiv \frac{3}{x^3}[\sin x - x\cos x] & (47)\\ \tilde{\rho_I} &\equiv \frac{3}{x^3}[1 + \tfrac{1}{2}x^2 - \cos x - x\sin x] \quad , & (48)\end{aligned}$$

provided $R_H = 1.9R_S$. (I have assumed that $\vec{q}$ is parallel to the symmetry axis of the hemisphere. It is left as an exercise to the student to determine if the same is true in the perpendicular case.) But since we don't have *a priori* knowledge of the values for R_H and R_S, it is extremely difficult to determine which is the preferred parameterization.

Three-particle interferometry, on the other hand, *in conjunction with the two-particle results*, can distinguish (in principle) between these two cases. The three-particle correlation function, defined as

$$C_3^{ideal}(\vec{p}_1,\vec{p}_2,\vec{p}_3) = \frac{\langle n\rangle^3}{\langle n(n-1)(n-2)\rangle} \cdot \frac{\dfrac{dn}{d\vec{p}_1 d\vec{p}_2 d\vec{p}_3}}{\dfrac{dn}{d\vec{p}_1}\dfrac{dn}{d\vec{p}_2}\dfrac{dn}{d\vec{p}_3}}\,, \tag{49}$$

reduces in the simplest case to

$$C_3(\vec{q}) = 1 + |\tilde{\rho}(\vec{q}_{12})|^2 + |\tilde{\rho}(\vec{q}_{23})|^2 + |\tilde{\rho}(\vec{q}_{31})|^2 + 2\mathbf{Re}[\tilde{\rho}(\vec{q}_{12})\tilde{\rho}(\vec{q}_{23})\tilde{\rho}(\vec{q}_{31})] \quad , \vec{q}_{ij} \equiv \vec{p}_i - \vec{p}_j \quad .(50)$$

It is precisely the term involving the real part of the source transforms that produces the sensitivity to asymmetric sources, since they necessarily have complex Fourier transforms.

The difficulty in using a three-fold correlation function lies in the choice of an appropriate relative momentum variable. (Recall that we had significant problems with only two particles!) Since this discussion is meant to be illustrative only, I have chosen to require $\vec{p}_1 = \vec{p}_2$. Then only $\vec{q}_{13}$ is non-zero, so the correlation function may be plotted versus this quantity (still assumed to be parallel to the symmetry axis of the hemisphere). The result, shown in Figure 6b for the same choice $R_H = 1.9R_S$, clearly shows that three-particle interferometry can resolve the shape differences to which two-particle interferometry is "blind". (Note that with the assumption that $\vec{p}_1 = \vec{p}_2$, the asymptotic value of $C_3(q)$ is 2, not 1.)

4 Coherence and Correlation Functions

It is not conventional to defer a discussion on coherence until this point in a monograph on interferometry. I have chosen to do so in order to emphasize that there are other concerns more directly related to the interpretation of experimental data. Nonetheless, the geometric information obtained from HBT depends essentially on an assumption concerning the coherence of the fields producing the quanta. Specifically, we find ourselves in the seemingly contradictory position of requiring that our interference "pattern" originate from a *incoherent* source. This is in contrast to ordinary=classroom=(Michelson or two-slit) interferometry, which requires that the wave be coherent.

The resolution to this "paradox" is familiar to any quantum optician: It is obvious that two-slit interferometry requires that there be a definite phase relationship between the wave incident on one slit and the wave incident on the other. We can arrange to do this either by putting many photons into one mode of the field (e.g., with a laser), or by guaranteeing that there is only one photon in the apparatus at any time. The interference, should we think of photons rather than waves, takes place "one photon at a time", and hence is often called first-order interference.

What I have called "interferometry" throughout this article is more properly described as "intensity interferometry" or "second-order interferometry". It depends in an essential way on the interference *between* (like) particles. In a sense, it measures the *fluctuations* resulting from the quantum interference of two particles. If one accepts this, it should then come as no surprise that the fluctuations are in some sense maximal when the source of particles is chaotic rather than incoherent.

To see precisely where the assumption of chaoticity appears in a simple calculation, I repeat here a derivation originally due to Bowler[22]. Roughly similar arguments have been presented by Gyulassy, Kaufmann and Wilson[37]. For the sake of definiteness, I will refer to pions. Assume that all pions originate from a current $J(p_1, x)$, where x

describes the spatial variation of the current over the source. The amplitude to detect a pion of momentum p_1 at infinity is then given by

$$A(p_1) = \int J(p_1, x) e^{ip_1 x}\, dx \quad , \tag{51}$$

and we assume that J is normalized such that the momentum distribution is given by

$$\frac{dn}{dp_1} = |A(p_1)|^2 \quad . \tag{52}$$

Similarly, the amplitude to detect two pions at infinity, one of momentum p_1, the other p_2, is

$$A(p_1, p_2) = A(p_1)A(p_2) \quad (!) \tag{53}$$

Note that this amplitude both integrates over all parts of the source *and* is symmetric under $p_1 \leftrightarrow p_2$! Then, evidently,

$$\frac{dn}{dp_1 dp_2} = |A(p_1, p_2)|^2 = |A(p_1)|^2 |A(p_2)|^2 \Rightarrow C_2(p_1, p_2) = 1 \quad (!), \tag{54}$$

that is, our correlation function is flat and we have thus have no sensitivity to the source size!

Have we made some foolish error in our quantum mechanics? Not at all. What we have derived is the correlation function for a *coherent* source, which implies that what is going on at x had a fixed phase with respect to what is going on at y. If this is *not* the case, as we would hope for a messy heavy ion collision, then we can transform the expressions in Eqs. 52 and 54:

$$\begin{aligned} |A(p_1)|^2 &= \int J^*(p_1, y) J(p_1, x) e^{ip_1(x-y)}\, dxdy & (55)\\ &= \int |J(p_1, x)|^2\, dx \qquad \textbf{if} \quad \langle J^*(p_1, y) J(p_1, x)\rangle \sim \delta(x-y) \quad . & (56) \end{aligned}$$

The second line in the above equation is at the heart of our assumption about an incoherent source. It states that not all parts of the source contribute to the physics at y (technically, that they are varying so fast with position that their net sum is essentially zero). For instance, in a heavy ion collision, one might identify J with the net current of Δ's,

$$J(p_1, x) = \sum_i \langle \pi \text{ with } p_1 \text{ at } x | \Delta_i \rangle \delta(x - x_i) \tag{57}$$

and assume that the decay at x_i is independent of the decay at x_j.

Proceeding for the two-particle case, we find that we must be more careful about the bookkeeping:

$$\begin{aligned} A(p_1, p_2)|^2 &= \int dydy'dxdx' J^*(p_1, y) J^*(p_2, y') J(p_1, x) J(p_2, x') e^{ip_1(x-y)+ip_2(x'-y')} & (58)\\ &= \int |J(p_1, x)|^2\, dx \cdot \int |J(p_2, x')|^2\, dx' & \\ &\quad + |\int J^*(p_2, y) J(p_1, x) e^{i(p_1-p_2)(x-y)}\, dxdy|^2 \quad , & (59) \end{aligned}$$

where again in the second step we have assumed that different parts of the source are mutually incoherent, *but the identical nature of the pions* requires that we must consider both the overlap of $(x = y; x' = y')$ *and* $(x = y', x' = y)$.

The steps to the usual correlation function should now be clear. While I have assumed that our quanta are pions, nothing in the derivation argues against any other boson. In fact, nothing in the derivation requires *particles*. For example, replacing $J(p_1, x) \rightarrow \vec{E}(k_1, x)$ would give a classical calculation of the fluctuations and correlation function expected for an electromagnetic field. Except for the averaging over the polarization, one obtains an HBT correlation indistinguishable from that expected from quantum mechanical considerations. Further discussion of the deep connection between waves, fluctuations, and coherence may be found in Refs. [38, 39, 40].

Experimentally, what evidence is there for coherence? The word most appropriate is "muddled". The crudest attempts to quantify the amount of coherence present in the correlation function depend on the deviations of the λ parameter from unity when fitting to the form $C_2(q) \equiv 1 + \lambda|\tilde{\rho}(q; R)|^2$. Unfortunately, λ is given by the intercept of $C_2 - 1$ as $q \rightarrow 0$, which of course is typically the least-populated region of C_2 (since the number of available pairs goes as $q^2\, dq$). Additionally, it is this region of q that is most sensitive to the two-particle Coulomb correction (assuming they are charged), and to the two-particle tracking-efficiency and momentum resolution. Even worse, precise knowledge of the momentum resolution is needed in order to correctly calculate the Coulomb correction.

Further complicating this is the contribution from resonance decays to the observed pions. This has *nothing* to do with the fact that pion was created by a resonance, and therefore its phase is "suspect". Pions originating from Δ and ρ decay are presumably perfectly suited to HBT measurements. The culprits are those pions created by the decays of η's, η''s, and to some extent ω's and K*'s, since these decays (largely) occur well outside any interaction region of size < 5 fm. For the heavy ion physicist, another complication results from the lack of data for resonance production in heavy ion collisions. One must use p-p or p-A data, and assume the ratio of direct to resonance production varies only slightly from these values as the mass of the colliding species increases.

Given these qualifications, it is not surprising that attempts to fit forms for $C_2(q)$ that specifically account for a coherent component[41] have met with both experimental[42] and phenomenological ambiguities[43]. There is a recent report[44] that the sensitivity of higher order correlation functions may overcome some of these problems. The extent to which these results are independent of the effects of resonances, and of the difficulties in parameterizing the relative momentum in higher order correlation functions remains an active area of investigation.

5 Caveats and Conclusions

In spite of my stated desire to provide a tutorial document on current interferometry, many topics have been neglected here. For instance, there is no discussion of the variations of radii with multiplicity, the actual experimental techniques for achieving high statistics for rare events have been ignored completely, final state interactions were mentioned only briefly– there is little need to go on, other than to acknowledge that there is much more to HBT than what is discussed here. On the other hand, the topics presented should be adequate to establish that interferometry is both powerful and subtle. Properly used, it will be of crucial importance in probing the complicated final states expected at RHIC. One does not need to be too much of an optimist to hope that by 1998 we will have available single-particle and correlation data for pions, kaons, protons, and perhaps anti-protons from Au+Au collisions at RHIC, along with a cascade code that correctly describes these results. Taken together, we can expect to have a *calibrated* model that allows us to investigate this exciting new regime of physics.

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7 APPENDIX: Review of Density Matrices

(This appendix is directly based on material found in Ref. [45] and Ref. [46].)

The density matrix is a natural method for describing a quantum mechanical system in which we have imperfect knowledge of some of its degrees of freedom. To begin, consider a nucleus described by an A body wavefunction $\Psi(x, y, \ldots, x_A)$. In this representation, we can write the expectation value of operator $\mathcal{O}(x)$ for particle '1' as

$$\langle \mathcal{O} \rangle = \int \Psi^*(x, x_2, \ldots, x_A)\mathcal{O}(x)\Psi(x, x_2, \ldots, x_A)\, d(x, x_2, \ldots, x_A) \equiv \int \rho(x, x)\mathcal{O}(x)\, dx \quad , \tag{60}$$

where the one-body density matrix $\rho(x, x')$ is given by

$$\rho(x, x') = \int \Psi^*(x, x_2, \ldots, x_A)\Psi(x', x_2, \ldots, x_A)\, d(x_2, \ldots, x_A) \quad . \tag{61}$$

Upon first exposure, one is typically puzzled by the insistence on writing the density matrix as a function of x and x'. However, consider the case where $\mathcal{O}(x) \equiv \nabla_1$. Then, the two-variable form for the density matrix lets us write with impunity

$$\langle \mathcal{O} \rangle = \int \mathcal{O}(x')\rho(x, x')\, dx'\, \delta(x - x')\, dx \quad , \tag{62}$$

without worrying about operator ordering.

In fact, what's going on here is something much deeper. First, note that Eq. 62 is a *trace* operation in the (continuous) coordinate representation:

$$\langle \mathcal{O} \rangle \equiv \mathrm{Tr}\{\rho\mathcal{O}\} \quad , \tag{63}$$

since (regarding both unlabeled ρ and $\mathcal{O}$ as operators)

$$\rho\mathcal{O} = \int \rho^{(1)}|x'\rangle\langle x'|\mathcal{O}\, dx' \quad , \tag{64}$$

so that

$$\mathrm{Tr}\{\rho\mathcal{O}\} \equiv \int \langle x|\rho\mathcal{O}|x\rangle\, dx = \int \langle x|\rho|x'\rangle\langle x'|\mathcal{O}|x\rangle\, dx' dx \quad , \tag{65}$$

which reduces to Eq. 62 for the case of a diagonal operator $\mathcal{O}$. This is more than a neat way of writing things; note that we are free to evaluate the trace in any representation that is convenient.

Secondly, although I have restricted this development to the case where ρ results from integrating the other coordinates of a pure state Ψ, this formalism can be extended in a straightforward way to include impure states. That is, if all we can say is that the system has probability $\mathcal{P}(i)$ to be in the pure state $|i\rangle$, it is still (very!) useful to define the density matrix via

$$\sum_i \mathcal{P}(i)\, |i\rangle\langle i| \quad . \tag{66}$$

An extremely useful form for the density matrix is the Wigner function, which can be motivated by the following argument. Given that the probability to find the system with momentum P is $\langle P|\rho|P\rangle \equiv \mathcal{P}(P)$, we can ask "Is there a representation (i.e., a function) $\langle R|\rho|P\rangle$ such that $\mathcal{P}(P) = \int dR\, \langle R|\rho|P\rangle$?" The positive answer "Yes" follows from just a few lines:

$$\begin{aligned}
\int dR\, \langle R|\rho|P\rangle &\equiv \langle P|\rho|P\rangle & (67)\\
&= \int dx\, dy\, \langle P|x\rangle\langle x|\rho|y\rangle\langle y|P\rangle & (68)\\
&= \int dR\, dr\, e^{iPr}\langle R + \tfrac{1}{2}r|\rho|R - \tfrac{1}{2}r\rangle \quad , & (69)
\end{aligned}$$

where in the last line I have assumed $\langle P|x\rangle$ and $\langle y|P\rangle$ are plane-wave states, and have transformed to relative coordinates. It is then apparent that

$$\rho_W(R,P) \equiv \langle R|\rho|P\rangle = \int dr\, e^{iPr}\langle R+\tfrac{1}{2}r|\rho|R-\tfrac{1}{2}r\rangle \quad , \tag{70}$$

where $\rho_W(R,P)$ is the Wigner function for the system. Given the complete equivalence of R and P in the above derivation, it should come as no surprise that this function has the properties of a classical phase-space function for R and P, that is, (switching to three-vector notation),

$$\mathcal{P}(\vec{P}) = \int f(\vec{P},\vec{R})d\vec{R} \tag{71}$$

$$\mathcal{P}(\vec{R}) = \int f(\vec{P},\vec{R})\frac{d\vec{P}}{(2\pi\hbar)^3} \quad . \tag{72}$$

Additionally, $\rho_W(\vec{R},\vec{P})$ can be used to calculate the expectation value of any operator that depends only on $\vec{P}$ or $\vec{R}$, for example,

$$\langle\mathcal{O}(\vec{R})\rangle = \int \mathcal{O}(\vec{R})\rho_W(\vec{P},\vec{R})\frac{d\vec{P}}{(2\pi\hbar)^3}d\vec{R} \quad . \tag{73}$$

However, this apparently classical behavior comes at a cost. We know that a.) Quantum mechanics forbids the simultaneous specification of a system's position and momentum and b.) It's not nice to fool Mother Nature. The particular revenge taken by Mother Nature is that $\rho_W(R,P)$ is not positive definite, which prohibits a one-to-one mapping between ρ_W and a truly classical phase space distribution function.

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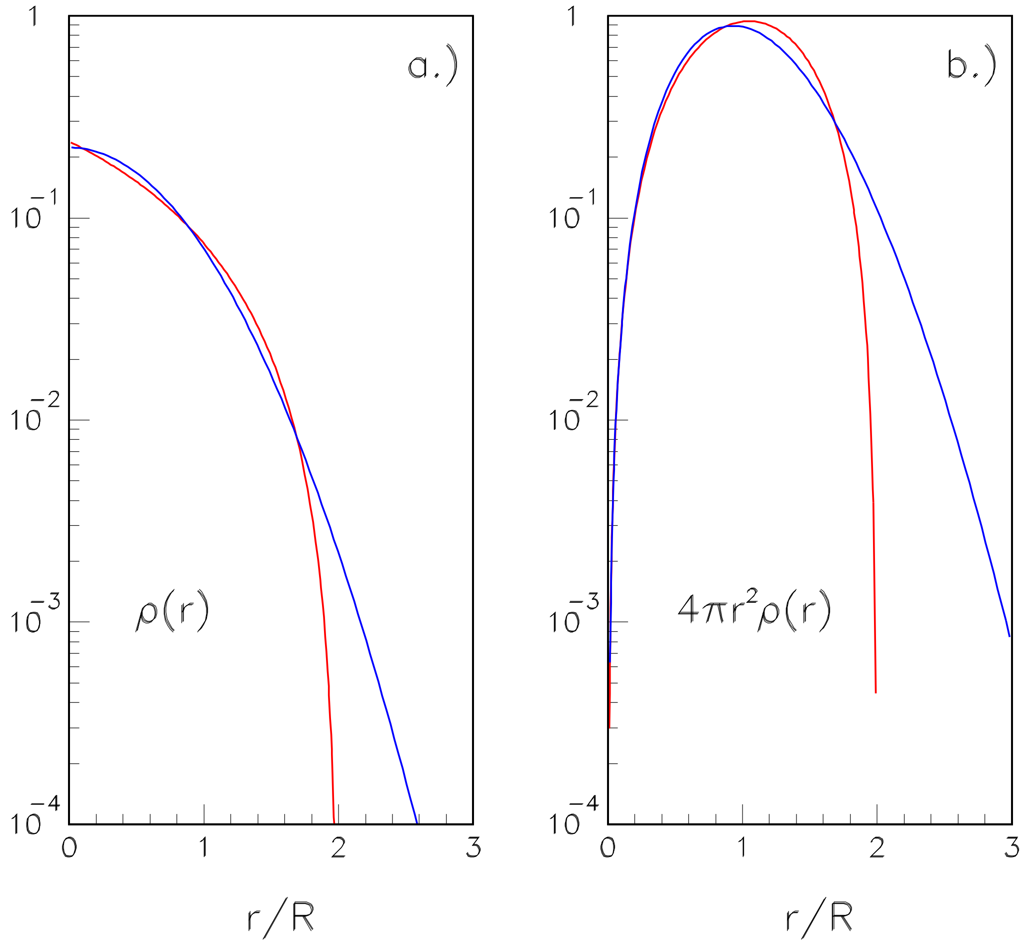

Figure 3: a.) The distribution $\rho(r)$, $r \equiv |\vec{r}_1 - \vec{r}_2|$, for a uniform sphere of radius R (solid lines), and a Gaussian distribution with rms radius $= 0.81R$ (dashed line). b.) As in a.), but for $\rho(r) \cdot 4\pi r^2$.

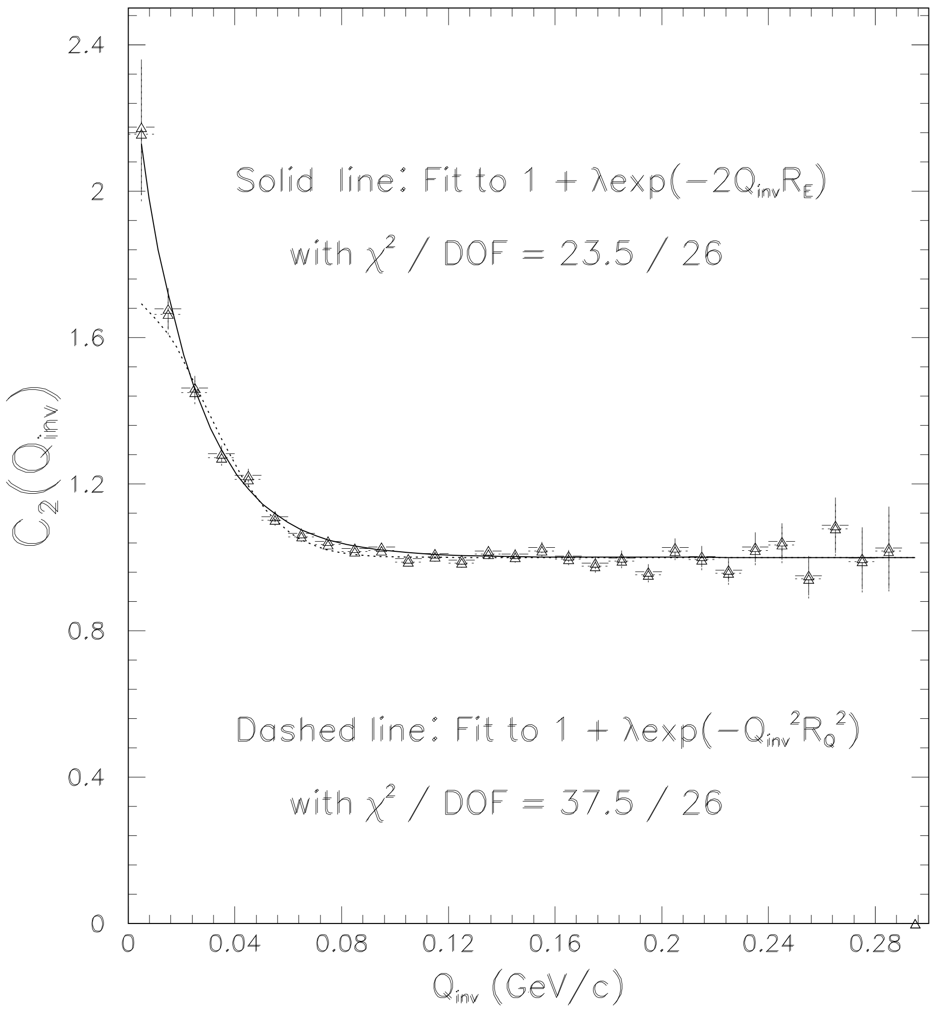

Figure 4: The result of projecting a spherically symmetric correlation function of the form given by Eq. 14 into the Q_{inv} variable. See text for discussion.

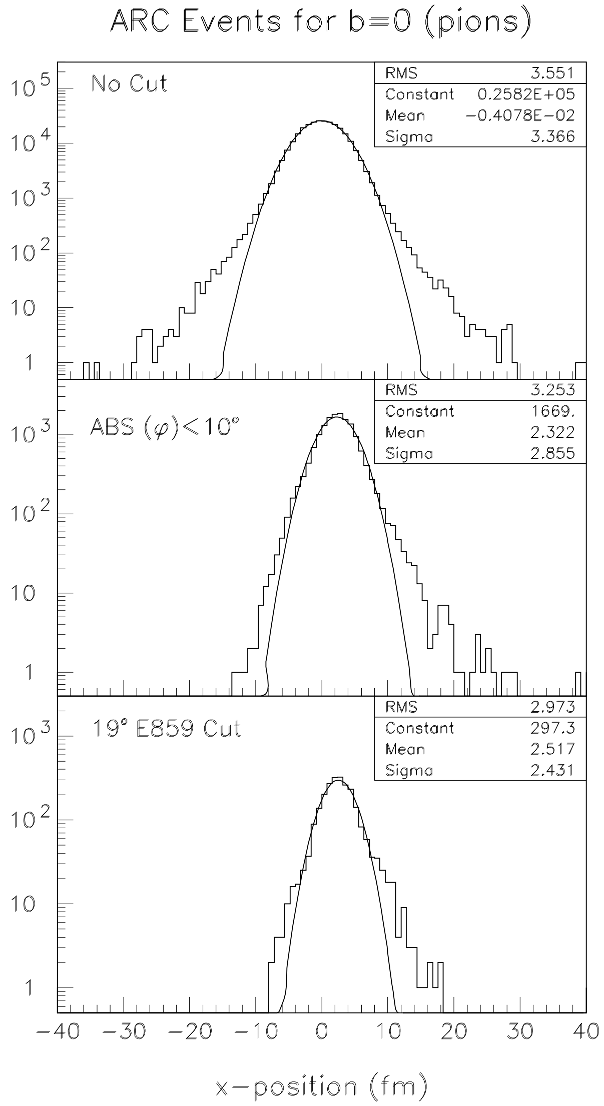

Figure 5: ARC predictions for the distribution of last strong interactions in transverse to the beam for central Si+Au collisions at AGS energies. The top panel shows the x distribution for all produced pions, the middle the result of imposing an acceptance cut in ϕ, while the bottom panel shows the result of applying the E859 acceptance in θ, ϕ, and momentum. For all three figures, the RMS value is for the entire distribution, while the mean and sigma are derived from Gaussian fits (solid lines) to the distributions.

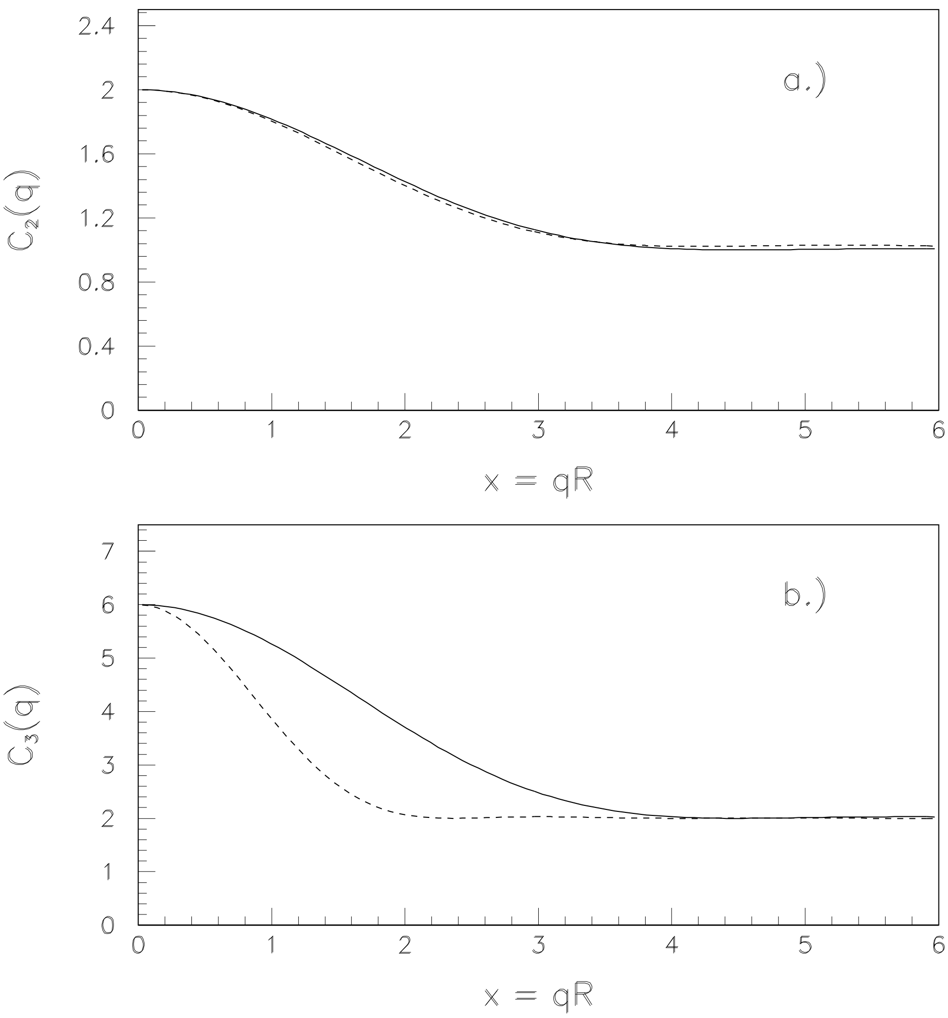

Figure 6: a.) The difference between the second order correlation function $C_2(q)$ for a uniform spherical distribution (solid line) and a uniform hemispherical distribution (dashed line) is shown versus the reduced variable $x \equiv qR$. The radius of the hemispherical volume is chosen to be $1.9R$, where R is the radius of the spherical volume. The direction of q is along the symmetry axis of the hemisphere. b.) As in a.), but for the three-pion correlation function $C_3(q)$, evaluated for the special case $\vec{p}_1 = \vec{p}_2 \neq \vec{p}_3$.